Regular and Semi-Regular 4D-Polytopes of the Coxeter-Weyl Group $W(F_4)$ and Quaternions

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The Coxeter-Weyl group $W(F_4)$ is constructed in terms of quaternions and its orbits representing the vertices of the 4D-polytopes corresponding to the Platonic and Archimedean versions of polyhedra in three dimensions are determined. The vertices of the polytopes are displayed in terms of discrete quaternions projected to three dimensions using its non-conjugate Coxeter-Weyl subgroups $W(SO(7))_L$ and $W(SO(7))_R$ which are the symmetries of the solids possessing octahedral symmetry. We also explain the cell structures of the 9 polytopes of interest and construct the vertices with the use of all subgroups $W(SO(7))_L$, $W(SO(7))_R$, $(D_3 \times Z_2)_L$, $(D_3 \times Z_2)_R$ of $W(F_4)$ acting in three dimensions.

1. Introduction

The Platonic solids have been used in Greek philosophy to classify fundamental matter: tetrahedron with fire, cube with earth, air with octahedron, and water with icosahedron. Kepler tried to explain the orbits of planets with Platonic solids placed in concentric spheres. Of course, these approaches were abandoned as science progressed. Recent discoveries have proved that Platonic solids and Archimedean solids, which are obtained from Platonic solids by rectifications and truncations, have been successfully used to explain the crystallographic structures in physics, molecular structures in chemistry [1] and virus capsids in biology [2].

The $O(4)$ symmetry of the 4D-Euclidean space can best be described by quaternions. In particular, the finite subgroups of $O(4)$ can be used to describe the Coxeter-Weyl groups of the Lie algebras $SO(8)$, $SO(9)$, F_4 [3] and the quasicrystallographic Coxeter group H_4 [4] where left-right multiplications of discrete quaternionic elements of the binary polyhedral groups are invoked. In the paper [5] we classified the Platonic and Archimeden solids in 3D-Euclidean space using the quaternionic descriptions of the orbits of the Coxeter-Weyl groups of $SU(4)$, $SO(7)$ and H_3. In a subsequent paper [6] we extended our work to determine the Coxeter-Weyl orbits of $W(SO(9))$ in terms of quaternions and classified the Platonic and Archimedean Polytopes of $W(SO(9))$.

In this paper we study the regular and semi-regular polytopes of $W(F_4)$ corresponding to Platonic and Archimedean versions of the 3D-polyhedra. In the paper [3] we constructed the root system of the exceptional Lie algebra F_4 in terms of quaternions and constructed the automorphism group $Aut(F_4) = \{[O,O] \oplus [O,O]^*\}$ in terms of the quaternionic elements of the binary octahedral group O. In what follows, using the Lie algebraic technique [7], we determine the quaternionic vertices of the regular and semi-regular polytopes of $W(F_4)$. In Section 2 we briefly summarize the quaternionic construction of $Aut(F_4) = \{[O,O] \oplus [O,O]^*\}$ and show how $W(SO(9))$ is embedded in $W(F_4)$ triply-symmetric way. We construct the rank-3 subgroups $W(SO(7))_L$, $W(SO(7))_R$, $(D_3 \times Z_2)_L$, $(D_3 \times Z_2)_R$ and discuss their embeddings in $W(F_4)$. In Section 3 we compute the orbit of $W(F_4)$ for an arbitrary highest weight $\Lambda = (a_1 a_2 a_3 a_4)$, with $(a_i \geq 0, i = 1,2,3,4)$ and give its decomposition under $W(SO(9))$. Section 4 is devoted to the constructions of regular and semi-regular polytopes and their cell structures where we summarize the results in Table 3.

2. Quaternionic representation of the Coxeter-Weyl group $W(F_4)$

Let $q = q_0 + q_i e_j$, $(i = 1,2,3)$ be a real unit quaternion with its conjugate defined by $\bar{q} = q_0 - q_i e_i$ and the norm $q\bar{q} = \bar{q}q = 1$. The quaternionic imaginary units satisfy the relations

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k, \ (i, j, k = 1, 2, 3) \qquad (1)$$

Here δ_{ij} and ϵ_{ijk} are the Kronecker and Levi-Civita symbols and summation over the repeated indices is implicit. With the definition of the scalar product

$$(p, q) = \frac{1}{2}(\bar{p}q + \bar{q}p), \qquad (2)$$

quaternions generate the four-dimensional Euclidean space. The group of quaternions is isomorphic to $SU(2)$ which is the double cover of the proper rotation group $SO(3)$. Its finite subgroups are classified as [8], infinite number of cyclic and infinite number of dicyclic groups in addition to the binary tetrahedral group T, binary octahedral group O and the binary icosahedral group I which are related to the ADE classification of the Lie algebras [9]. An orthogonal rotation in 4D-Euclidean space can be represented by the group elements of $O(4)$ [4] as

$$[a, b] : q \rightarrow q' = aqb, [c, d]^* : q \rightarrow q'' = c\bar{q}d, \qquad (3)$$

where a, b, c, d are unit quaternions and q can be a quaternion with arbitrary norm. The finite subgroups of $O(4)$ follows the finite subgroups of $SU(2)$. The relevant finite subgroup of $SU(2)$ here is the binary octahedral group O which can be decomposed as follows:

$$O = T \oplus T'. \qquad (4)$$

Here T represents the binary tetrahedral group given by

$$T = \{\pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)\}, \ (5)$$

and

$$\begin{aligned} T' \ &= \ \{\frac{1}{\sqrt{2}}(\pm 1 \pm e_1), \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3), \\ &\quad \frac{1}{\sqrt{2}}(\pm 1 \pm e_2), \frac{1}{\sqrt{2}}(\pm e_3 \pm e_1), \\ &\quad \frac{1}{\sqrt{2}}(\pm 1 \pm e_3), \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2)\}. \end{aligned} \qquad (6)$$

They can also be put in the form

$$T = \sum_{a=0}^{3} \{\pm e_a \oplus \pm \omega_a \oplus \pm \bar{\omega}_a\} \qquad (7)$$

$$\begin{aligned} T' \ &= \ \sum_{a \neq b = 0}^{3} \oplus \frac{1}{\sqrt{2}} \{\pm e_a \pm e_b\} \\ &= \ \sum_{a \neq b = 0}^{3} \oplus \frac{1}{\sqrt{2}} \{\pm \omega_a \pm \omega_b\} \\ &= \ \sum_{a \neq b = 0}^{3} \oplus \frac{1}{\sqrt{2}} \{\pm \bar{\omega}_a \pm \bar{\omega}_b\} \end{aligned} \qquad (8)$$

Here we define

$$\begin{aligned} \omega_0 \ &= \ \frac{1}{2}\{1 + e_1 + e_2 + e_3\}, \\ \omega_1 \ &= \ \frac{1}{2}\{1 + e_1 - e_2 - e_3\} \\ \omega_2 \ &= \ \frac{1}{2}\{1 - e_1 + e_2 - e_3\}, \\ \omega_3 \ &= \ \frac{1}{2}\{1 - e_1 - e_2 + e_3\}. \end{aligned} \qquad (9)$$

Let $p, q \in O$ be arbitrary elements of the binary octahedral group, then the set of elements

$$Aut(F_4) \approx W(F_4) : Z_2 = \{[p, q] \oplus [p, q]^*\} \qquad (10)$$

is the extension of the Coxeter-Weyl group $W(F_4)$ by the diagram symmetry [3] as shown in Fig. 1. The group structure in (10) follows from the group generators obtained from Fig. 1.

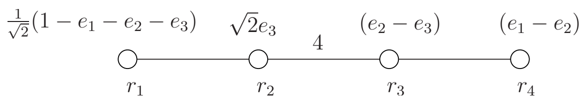


FIG. 1: The Coxeter-Dynkin diagram of $W(F_4)$.

In general, the reflection generator r of an arbitrary Coxeter group with respect to a hyperplane represented by the vector α is given by the action

$$r : \Lambda \rightarrow \Lambda - \frac{2(\Lambda, \alpha)}{(\alpha, \alpha)}\alpha. \qquad (11)$$

When Λ and α are represented by quaternions the equation (8) reads

$$r : \Lambda \rightarrow -\frac{\alpha\bar{\Lambda}\alpha}{(\alpha, \alpha)},$$

and the generators of $W(F_4)$ would be given by

$$\begin{aligned} r_1 &= [\frac{1}{2}(1-e_1-e_2-e_3), \frac{1}{2}(-1+e_1+e_2+e_3)]^*, \\ r_2 &= [e_3, -e_3]^*, \\ r_3 &= \frac{1}{2}[(e_2-e_3), (-e_2+e_3)]^*, \\ r_4 &= \frac{1}{2}[(e_1-e_2), (-e_1+e_2)]^*. \end{aligned} \quad (12)$$

To represent the group elements of $W(F_4)$ in terms of the elements of the binary octahedral group O and identify its subgroup $W(SO(9))$ easily we introduce the subsets of O defined by [3]

$$O = T \oplus T' = \{V_0 \oplus V_+ \oplus V_-\} \oplus \{V_1 \oplus V_2 \oplus V_3\} \quad (13)$$

with

$$\begin{aligned} V_0 &= \sum_{a=0}^{3} \oplus \pm e_a, \ V_+ = \sum_{a=0}^{3} \oplus \pm \omega_a, \ \bar{V}_- \\ &= \sum_{a=0}^{3} \oplus \pm \bar{\omega}_a, \\ V_1 &= \{\frac{1}{\sqrt{2}}(\pm 1 \pm e_1)\}, \ \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3)\}, \\ V_2 &= \{\frac{1}{\sqrt{2}}(\pm 1 \pm e_2)\}, \ \frac{1}{\sqrt{2}}(\pm e_3 \pm e_1)\}, \\ V_3 &= \{\frac{1}{\sqrt{2}}(\pm 1 \pm e_3)\}, \ \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2)\}. \end{aligned} \quad (14)$$

They satisfy the multiplication table given in Table I.

TABLE I: Multiplication table of the binary octahedral group.

	V_0	V_+	V_-	V_1	V_2	V_3
V_0	V_0	V_+	V_-	V_1	V_2	V_3
V_+	V_+	V_-	V_0	V_3	V_1	V_2
V_-	V_-	V_0	V_+	V_2	V_3	V_1
V_1	V_1	V_2	V_3	V_0	V_+	V_-
V_2	V_2	V_3	V_1	V_-	V_0	V_+
V_3	V_3	V_1	V_2	V_+	V_-	V_0

It was shown in [3] that the elements of $W(F_4)$ can be written as follows

$$W(F_4) \approx A \oplus A^*. \quad (15)$$

where

$$\begin{aligned} A &= [T,T] \oplus [T',T'] \\ &= \{\sum_{a,b} \oplus [V_a, V_b] \oplus \sum_{i,j} \oplus [V_i, V_j]\}, \\ A^* &= [T,T]^* \oplus [T',T']^* \\ &= \{\sum_{a,b} \oplus [V_a, V_b]^* \oplus \sum_{i,j} \oplus [V_i, V_j]^*\}, \\ & a,b = 0,+,-; i,j = 1,2,3. \end{aligned} \quad (16)$$

One notes that the number of elements in (15) is 1152. The Dynkin diagram symmetry implies that

$$\begin{aligned} 1 &\rightarrow \frac{1}{\sqrt{2}}(1+e_1), \ e_1 \rightarrow \frac{1}{\sqrt{2}}(1-e_1) \\ e_2 &\rightarrow \frac{1}{\sqrt{2}}(e_2+e_3), \ e_3 \rightarrow \frac{1}{\sqrt{2}}(e_2-e_3) \end{aligned} \quad (17)$$

which can be generated by

$$Z_2 = [\frac{1}{\sqrt{2}}(e_2+e_3), -e_2]. \quad (18)$$

A left or right multiplication of $W(F_4)$ by Z_2 would extend the group $W(F_4)$ to the group $Aut(F_4) : Z_2$. The group $W(SO(9))$ of order 384 is a maximal subgroup of $W(F_4)$ with an index 3. It can be represented as

$$W(SO(9)) \approx B \oplus C \oplus B^* \oplus C^* \quad (19)$$

with

$$\begin{aligned} B &= \{[V_0, V_0] \oplus [V_+, V_-] \oplus [V_-, V_+]\}, \\ C &= \{[V_1, V_1] \oplus [V_2, V_2] \oplus [V_3, V_3]\}. \end{aligned} \quad (20)$$

The group $W(F_4)$ can be expressed as the sum of sets

$$\begin{aligned} W(F_4) &= \sum_{i=0}^{2} \oplus W(SO(9)) d^i \\ &= W(SO(9)) \oplus W(SO(9)) d \oplus W(SO(9)) d^2 \end{aligned} \quad (21)$$

where the coset representative can be chosen as an arbitrary element of $d \in [V_+, V_0]$, say $d = [\omega_0, 1]$. The three conjugate groups of $W(SO(9))$ in $W(F_4)$ can be written as

$$\begin{aligned} & W(SO(9)), \ [\omega_0, 1] W(SO(9)) [\bar{\omega}_0, 1], \\ & [\bar{\omega}_0, 1] W(SO(9)) [\omega_0, 1]. \end{aligned} \quad (22)$$

The two subgroups $W(SO(7))_L$ and $W(SO(7))_R$ can be generated by the set of generators (r_1, r_2, r_3)

and (r_2, r_3, r_4) respectively. Each $W(SO(7))$ can be embedded in the group $W(F_4)$ in 24 different ways as its 24 conjugates. However, $W(SO(7))_L$ and $W(SO(7))_R$ are not conjugate to each other in $W(F_4)$. We will show that the conjugacy between them will be restored when they are embedded in $Aut(F_4)$. The prismatic groups $(D_3 \times Z_2)_L$ and $(D_3 \times Z_2)_R$ are generated by (r_1, r_2, r_4) and (r_1, r_3, r_4), respectively, and they have similar conjugacy properties as $W(SO(7))$ subgroups. Each $(D_3 \times Z_2)$ has an index 96 in $W(F_4)$ which counts the number of triangular prismatic cells of the given type. To work out the orbit structure of $W(F_4)$, we use a simpler representation of $W(SO(9))$ that can be written as the semi-direct product (:) of the elementary abelian group $Z_2^4 = Z_2 \times Z_2 \times Z_2 \times Z_2$ and the symmetric group S_4, that is, $W(SO(9)) \approx Z_2^4 : S_4$ [6]. The generators of Z_2^4 and S_4 can be taken respectively as

$$[1, -1]^*,\ [e_1, -e_1]^*,\ [e_2, -e_2]^*\ [e_3, -e_3]^* \quad (23)$$

and

$$\begin{aligned} a &= \frac{1}{2}[(1+e_2)],\ \omega_0(1-e_2)\omega_0]^*,\ a^4 = [1,1], \\ b &= [\omega_0, \bar{\omega}_0],\ b^3 = [1,1]. \end{aligned} \quad (24)$$

Here the generators a, b of S_4 permute the generators of the elementary abelian group Z_2^4 by conjugation, therefore, Z_2^4 is an invariant subgroup of the group $W(SO(9))$ and the elements of S_4 can be compactly written as [5, 6]

$$S_4 = \{[p, \bar{\omega}_0 \bar{p} \omega_0] \oplus [t, \omega_0 \bar{t} \omega_0]^*\},\ p \in T,\ t \in T'. \quad (25)$$

It is isomorphic to the Coxeter-Weyl subgroup $W(SU(4))$ of $W(SO(9))$ leaving the quaternion ω_0 invariant. The generators r_2, r_3, r_4 generate the subgroup

$W(SO(7))_R \approx S_4 \times Z_2$ given by [5, 6]

$$\begin{aligned} &W(SO(7))_R \approx S_4 \times Z_2 \\ &\quad = \{[p, \bar{p}] \oplus [t, \bar{t}] \oplus [p, \bar{p}]^* \oplus [t, \bar{t}]^*\} \end{aligned} \quad (26)$$

leaving the real unit quaternion $1 \in T$ invariant. As for the other conjugate groups of $W(SO(7))_R$, each fixes one element of T. Similarly, the subgroup $W(SO(7))_L \approx S_4 \times Z_2$ generated by r_1, r_2, r_3 can be represented by

$$\begin{aligned} &W(SO(7))_L \approx S_4 \times Z_2 \\ &\quad = \{[p, \bar{c}\bar{p}c] \oplus [t, \bar{c}\bar{t}c] \oplus [p, c\bar{p}c]^* \oplus [t, c\bar{t}c]^*\} \end{aligned} \quad (27)$$

leaving $c = \frac{1}{\sqrt{2}}(1 + e_1) \in T'$ invariant, implying that the conjugates of $W(SO(7))_L$ leave the elements of T' invariant. Group structures of the left-right prismatic groups will be discussed when we study the cell structures of the relevant polytopes.

3. Orbit of the group $W(F_4)$ in terms of quaternions

Let $\Lambda = (a_1\ a_2\ a_3\ a_4)$, $(a_i \geq 0,\ i = 1, 2, 3, 4)$ represents a vector with positive integer Dynkin labels [7] which characterizes the irreducible representations of the Lie group $W(F_4)$. Although the simple roots in Fig. 1 are not representing the true roots of the Lie algebra F_4 since short roots are converted to long roots, the same Lie algebraic technique can be used to determine the Coxeter-Weyl orbits, namely, one can determine the vertices of the polytopes possessing $W(F_4)$ symmetry using the highest weight technique with $\Lambda = (a_1\ a_2\ a_3\ a_4)$, $(a_i \geq 0,\ i = 1, 2, 3, 4)$. We note in passing that the Dynkin diagram symmetry implies $(a_1\ a_2\ a_3\ a_4) \to (a_4\ a_3\ a_2\ a_1)$. The highest weight vector can be written as a linear combination of the simple roots of Fig. 1,

$$\begin{aligned} \Lambda &= x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4 \\ &\qquad \alpha_4 = (x_1, x_2, x_3, x_4)^T C \end{aligned} \quad (28)$$

where C is the Cartan matrix of F_4 with short roots replaced by long roots,

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \quad (29)$$

Equation (28) determines the vector Λ in terms of quaternionic units in the form $\Lambda^{(0)} = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ where the coefficients of the unit quaternions are given by

$$\begin{aligned} \alpha_0 &= \sqrt{2}(a_1 + \frac{3}{2}a_2) + 2a_3 + a_4, \\ \alpha_1 &= \frac{1}{\sqrt{2}}a_2 + a_3 + a_4, \\ \alpha_2 &= \frac{1}{\sqrt{2}}a_2 + a_3, \\ \alpha_3 &= \frac{1}{\sqrt{2}}a_4. \end{aligned} \quad (30)$$

We can construct three weight vectors from $\Lambda^{(0)}$ each transforming under the same group

$W(SO(9))$ represented by (19). This can be achieved by applying $W(F_4)$ given in (21) on the highest weight $\Lambda^{(0)}$. Leading to the weight vectors of the group $W(SO(9))$:

$$\begin{aligned}\Lambda^{(0)} &= \alpha_0+\alpha_1e_1+\alpha_2e_2+\alpha_3e_3,\\ \Lambda^{(1)} &= \omega_0\Lambda^{(0)}=\alpha_0\omega_0-\alpha_2\omega_1-\alpha_3\omega_2-\alpha_1\omega_3\\ &= \beta_0+\beta_1e_1+\beta_2e_2+\beta_3e_3, \qquad (31)\\ \Lambda^{(2)} &= \omega_0^2\Lambda^{(0)}=-\bar{\omega}_0\Lambda^{(0)}\\ &= -\alpha_0\bar{\omega}_0-\alpha_3\bar{\omega}_1-\alpha_1\bar{\omega}_2-\alpha_2\bar{\omega}_3,\\ &= \gamma_0+\gamma_1e_1+\gamma_2e_2+\gamma_3e_3.\end{aligned}$$

Here β_a and γ_a are some linear combinations of α_a and thereof $a_i, i=1,2,3,4$. When we apply the Coxeter-Weyl group $W(SO(9))$ expressed in equations (19) and (20) on the vectors given by (31) we obtain 3 sets of $W(SO(9))$ orbits. For general vectors, where the integers $a_i, i=1,2,3,4$ take non-zero values, each orbit has a size of 384 which leads to the $W(F_4)$ orbit of size $384\times 3=1152$. To convince the reader that the group $W(SO(9))$ has an orbit of largest size 384 [6], we give the following procedure for the construction of the $W(SO(9))$ orbits in terms of quaternionic weights. Since the generators of the abelian group Z_2^4 change the signs of the quaternionic units $1, e_1, e_2, e_3$ its action on a vector say $\Lambda^{(0)}$ would lead to the set of 16 quaternions

$$\pm\alpha_0\pm\alpha_1e_1\pm\alpha_2e_2\pm\alpha_3e_3. \qquad (32)$$

The group S_4 represented by the generators in (19) simply permutes the quaternionic units $1, e_1, e_2, e_3$ when acting on (32) so that the size of each $W(SO(9))$ orbit will be $16\times 24=384$. Repeating the same procedure on the other two vectors $\Lambda^{(1)}$ and $\Lambda^{(2)}$ will yield to all vectors of the orbit $W(F_4)$. In the next section, we will discuss the structures of the regular and semi-regular orbits in terms of quaternions.
Before we proceed further, a few more facts are in order. Any subgroup acting in 3D-Euclidean subspace leaves one of the vector invariant as we have pointed out in the case of $W(SO(7))_L$ and $W(SO(7))_R$ leaving $c=\frac{1}{\sqrt{2}}(1+e_1)\in T'$ and $1\in T$ invariant, respectively. In what follows we prove two lemmas regarding the orbits of the subgroups acting in 3D.

Lemma 1: All the vectors in an orbit of $H\subset W(F_4)$, where H is a subgroup acting in 3D-Euclidean space will have the same scalar product with the vector q fixed by the subgroup H.

Proof: Let h be any quaternion in one of the orbit of H. Let the scalar product between q and h be given by $(q,h)=\frac{1}{2}(\bar{q}h+\bar{h}q)=f$, where f is a scalar. Then it is straightforward to show that $(h',q)=f$ where $h'=Hh$. Therefore, the classifications of the H-orbits in the orbits of $W(F_4)$ are the classifications of the scalar values f.

Lemma 2: Let the set of quaternions Q represent the vertices of a regular or a semi-regular polyhedra of $W(SO(7))_R$ within one of the orbit of $W(F_4)$, where q is fixed by $W(SO(7))_R$. Then, $rQ,\ r\in T$ represents the set of vertices of the of the same regular or semi-regular polyhedra transforming under the conjugate group $W'(SO(7))_R$ fixing the quaternion rq.

Proof: The proof is straightforward because the scalar product $(Q,q)=f$ is left invariant under a left multiplication of quaternions, i.e., $(rQ,rq)=f$. This is also true for a right multiplication of the quaternionic vertices by the elements of T. This fact allows us to count the number of cells of $W(SO(7))_R$ in $W(F_4)$ which is 24 and determine all the vertices of the $W(F_4)$ orbit by quaternionic multiplication. For the orbits of $W(SO(7))_L$, we have to take the quaternions $r\in T'$ to determine the vertices of the $W(SO(7))_L$ cells. Similar arguments are valid for the prismatic groups, but the choice of the quaternions r are not as straight as it is here. These cases will be discussed in the next section when the relevant cell structure is of interest. However, we would like to discuss the criteria about the existence of the regular prismatic cells in the orbits of the group $W(F_4)$.

Cells of the prismatic group $(D_3\times Z_2)_R$
It is generated by r_1, r_3, r_4 and acts on the quaternionic units as follows:

$$\begin{aligned}r_3r_4 &= [\bar{\omega}_0,\omega_0]:1\rightarrow 1,\\ &\qquad e_1\rightarrow e_3,\ e_2\rightarrow e_1,\ e_3\rightarrow e_2,\\ r_1 &= [\bar{\omega}_0,-\bar{\omega}_0]^*:1\rightarrow\omega_0,\\ &\qquad e_1\rightarrow\omega_1,\ e_2\rightarrow\omega_2,\ e_3\rightarrow\omega_3. \qquad (33)\end{aligned}$$

It is easy to check that the group $(D_3\times Z_2)_R$ fixes the quaternion $3+e_1+e_2+e_3=2(1+\omega_0)$. Since the dihedral group $(D_3)_R$ is a subgroup of the group $W(SO(7))_R$, one can find other generators like $(r_3r_4)^{(a)}=[\bar{\omega}_a,\omega_a]$ in the group $W(SO(7))_R$ and $r_1^{(a)}=[\bar{\omega}_a,-\bar{\omega}_a]$; $a=0,1,2,3$ to determine the conjugate groups of $(D_3\times Z_2)_R$ without changing the generators of the group $W(SO(7))_R$. New copies of $(D_3\times Z_2)_R$ would yield further invariants resulting in $3 \pm e_1 \pm e_2 \pm e_3$ (even (-) sign). Multiplying this set of vectors by the elements of T would lead to 96 quaternions which are fixed by the conjugates of $(D_3 \times Z_2)_R$ in $W(F_4)$. Using Lemma 2 one can find the vertices of the 96 triangular prisms by multiplying the vertices of one prismatic cell by 96 unit quaternions.

Now we explain in which orbits the prismatic cells occur. Let us apply the action of elements of $(D_3 \times Z_2)_R$ in (33) on the highest weight $\Lambda^{(0)} = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ to create 6 vertices where the coefficients $\alpha_a,\ a = 0, 1, 2, 3$ are given in (30):

$$\begin{aligned} &\alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \\ &\alpha_0 + \alpha_2 e_1 + \alpha_3 e_2 + \alpha_1 e_3, \\ &\alpha_0 + \alpha_3 e_1 + \alpha_1 e_2 + \alpha_2 e_3, \\ &\alpha_0\omega_0 + \alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3, \\ &\alpha_0\omega_0 + \alpha_2\omega_1 + \alpha_3\omega_2 + \alpha_1\omega_3, \\ &\alpha_0\omega_0 + \alpha_3\omega_1 + \alpha_1\omega_2 + \alpha_2\omega_3. \end{aligned} \tag{34}$$

The first three and last three quaternions of (34) each represents the vertices of triangles cyclicly permuted by the group element $r_3 r_4$. The edges of the triangles are represented by the quaternions in the cyclic order

$$\begin{aligned} &a_4 e_1 + a_3 e_2 - (a_4 + a_3) e_3, \\ &a_3 e_1 - (a_3 + a_4) e_2 + a_4 e_3, \\ &-(a_3 + a_4) e_1 + a_4 e_2 + a_3 e_3. \end{aligned} \tag{35}$$

In order the quaternions in (35) represent the edges of an equilateral triangle the Dynkin labels should satisfy the equation

$$a_3^2 + a_4^2 + a_3 a_4 = 1 \tag{36}$$

with the edge length $\sqrt{2}$. (36) is satisfied for either $a_3 = 1, a_4 = 0$ or $a_3 = 0, a_4 = 1$. Two parallel equilateral triangles are connected with the lines parallel to the quaternion $\frac{a_1}{\sqrt{2}}(1 - e_1 - e_2 - e_3)$. This would represent an edge of the triangular prism with the length $\sqrt{2}$ if $a_1 = 1$. Consequently, the highest weights of $W(F_4)$, which will involve triangular prisms, would be of the form $(1a_2 01)$ and $(1a_2 10)$ where the label a_2 can take any values $0, 1, 2, 3, \ldots$. But we will restrict our work with $a_2 = 0, 1$ because regular and semi-regular polytopes are obtained only from the highest weights with the Dynkin labels $a_i = 0, 1;\ i = 1, 2, 3, 4$.

Cells of the prismatic group $(D_3 \times Z_2)_L$
This subgroup generated by r_1, r_2, r_4 has the following actions on the quaternionic units

$$\begin{aligned} r_1 r_2 &: 1 \to \omega_0,\ e_1 \to \omega_1,\ e_2 \to \omega_2,\ e_3 \to -\omega_3 \\ r_4 &: 1 \to 1,\ e_1 \to e_2,\ e_2 \to e_1,\ e_3 \to e_3. \end{aligned} \tag{37}$$

It leaves the quaternion $2 + e_1 + e_2$ invariant. The same invariant vector can be also obtained from the invariant $3 + e_1 + e_2 + e_3 = 2(1 + \omega_0)$ by using the Dynkin diagram symmetry because the group $(D_3 \times Z_2)_L$ is transformed to the group $(D_3 \times Z_2)_R$ under the F_4 diagram symmetry. Therefore, the vertices of the prisms of $(D_3 \times Z_2)_L$ can be obtained from those of $(D_3 \times Z_2)_R$ by letting $1 \to \frac{1}{\sqrt{2}}(1 + e_1)$, $e_1 \to \frac{1}{\sqrt{2}}(1 - e_1)$, $e_2 \to \frac{1}{\sqrt{2}}(e_2 + e_3)$, $e_3 \to \frac{1}{\sqrt{2}}(e_2 - e_3)$. The Dynkin diagram symmetry would lead to the highest weights $(10a_2 1)$ and $(01a_2 1)$ possessing the triangular prism, vertices of which are transformed under the prism, group $(D_3 \times Z_2)_L$.

4. The $W(F_4)$ orbits as regular and semi-regular polytopes

Now, we can determine the vertices of the regular and semi-regular polytopes in terms of quaternions and study their cell structures. We note that the regular and the semi-regular polytopes of any Coxeter-Weyl group are obtained by assigning to the Dynkin labels the integers 0 and 1 only. The weights involving Dynkin labels with $a_i \geq 2$ represent orbits corresponding to the polytopes possessing the same symmetry however edges, faces, cells etc. are not regular.

Regular Polytope {3, 4, 3}-(24-cell) as $W(F_4)$ orbit-(1000) and orbit-(0001)
The Coxeter-Weyl group $W(F_4)$ has only one regular polytope denoted by the Schlaffli symbol {3, 4, 3} which is self-dual. First, let us study the orbit-(1000) because the other will follow from the Dynkin diagram symmetry. If one substitutes $a_1 = 1,\ a_2 = a_3 = a_4 = 0$ in (31) and applies the action of the group $W(SO(9))$ on the weights

$$\Lambda^{(0)} = \sqrt{2},\ \Lambda^{(1)} = \sqrt{2}\omega_0,\ \Lambda^{(2)} = -\sqrt{2}\bar{\omega}_0 \tag{38}$$

$\Lambda^{(0)}$ leads to the set of quaternions $\sqrt{2}V_0$ and the other two would generate the same set of quaternions $\sqrt{2}(V_+ \oplus V_-)$. Therefore, they altogether represent the set of quaternions $\sqrt{2}T$ where T is the binary tetrahedral group given by the quaternions in (5)and (7). That is to say, the 24-cell decomposes under $W(SO(9))$ as 24=8+16. The Dynkin diagram symmetry allows us to compute the vertices of the orbit-(0001) leading to the quaterions $\sqrt{2}T'$ where T' is defined in (6) and (8). Since T and T' are transformed into each other under the Dynkin diagram symmetry represented by $Z_2 = [\frac{1}{\sqrt{2}}(e_2 + e_3), -e_2]$ in (18) both orbits represent the same regular polytope, and are simply rotated with respect to each other. That is why it is called a self-dual polytope {3, 4, 3}. We will work with one of these and investigate its cell structure. We will work in the set $\sqrt{2}T'$, where the set of 6 quaternions

$$1 \pm e_1,\ 1 \pm e_2,\ 1 \pm e_3 \qquad (39)$$

transform as the vertices of an octahedron under the group $W(SO(7))_R$ defined by (26). This fixes the quaternion 1, but otherwise permutes the set of quaternions $\pm e_1, \pm e_2, \pm e_3$ representing the vertices of an octahedron in the 3D-space orthogonal to the quaternion 1. Of course, the quaternions $-1 \pm e_1,\ -1 \pm e_2,\ -1 \pm e_3$ in 4D-space represent another octahedron which is fixed by the same group in (26). As we have discussed above, the group $W(SO(7))_R$ can be embedded in 24 different ways in $W(F_4)$ and where one of the quaternion of T is fixed by the group $W(SO(7))_R$. Therefore, we obtain in each case a set of six vectors which represent the vertices of an octahedron. For example, let us consider the group $W(SO(7))_R$ that leaves $\omega_0 = \frac{1}{2}(1 + e_1 + e_2 + e_3) \in T$ invariant. Then, by Lemma 2, a left or right multiplication of the set of quaternions in (39) would give us the vertices of the new octahedron

$$\begin{aligned} &\omega_0\{1 \pm e_1,\ 1 \pm e_2,\ 1 \pm e_3\} \\ &\quad = \{1 \pm e_1,\ 1 \pm e_2,\ 1 \pm e_3\}\omega_0 \\ &\quad = \{\omega_0 \pm \omega_1,\ \omega_0 \pm \omega_2,\ \omega_0 \pm \omega_3\} \qquad (40) \end{aligned}$$

in the space generated by the unit vectors $\omega_1, \omega_2, \omega_3$. This proves that the orbit-(0001) consists of 24 octahedral cells with 24 vertices. The center of this octahedron is ω_0 up to a scale factor which is one of the vertices of the polytope represented by the orbit-(1000). Actually, the quaternion fixed by the group $W(SO(7))_R$ represents the center of the polyhedron left invariant by the group $W(SO(7))_L$. One can obtain the number of triangular faces from the index $\frac{|W(F_4|}{|D_3 \times Z_2|} = 96$. The number of edges can be determined from the argument that there exists 8 nearest vertices to a given vertex which leads to the $4 \times 24 = 96$ edges. The Euler characteristic of the 4D polytopes $\kappa = V - E + F - C = 0$, where V, E, F, C are the number of vertices, number of edges, number of faces and number of cells, respectively, is satisfied by all 4D-polytopes [8].

Semi-regular polytopes of $W(F_4)$ symmetry

There are 9 semi-regular polytopes left invariant by the group $W(F_4)$. The polytopes here have equal edge lengths but can be constructed from different regular cells. Only 8 of them can be obtained using the representation technique of Lie algebra. The 9$^{\text{th}}$, called the *snub* 24-*cell*, cannot be derived with the technique discussed here. Three of the 8 semi-regular polytopes (Archimedean polytopes) are symmetric under the Dynkin diagram symmetry and have larger symmetry $Aut(F_4)$. Now, in turn, we discuss how one determines the vertices of each semi-regular polytope and study its cell structure.

The orbit-(0010) (rectified 24-cell)

It has the same structure with the orbit-(0100) because of the diagram symmetry. Therefore, we will study only one of them, i.e., the orbit-(0010).This orbit is represented by the highest weight $\Lambda^{(0)} = 2 + e_1 + e_2$. One can apply the generators of $W(F_4)$ and generate the 96 quaternionic vertices of polytope which decomposes under $W(SO(9))$ as $96 = 64 + 32$. We will not reproduce them here, rather we will discuss their cell structure. This polytope does not possess the prismatic cells as we have discussed above. We will check now the regular or semi-regular orbits of $W(SO(7))_R$ and $W(SO(7))_L$ in the orbit-(0010). When we apply the generators of $W(SO(7))_R$ of (26), we obtain 12 quaternions given by

$$Q = \{2 \pm e_1 \pm e_2,\ 2 \pm e_2 \pm e_3, 2 \pm e_3 \pm e_1\}. \quad (41)$$

We note that the scalar part 2 of the quaternions is fixed by the group $W(SO(7))_R$. These vertices represent a *cuboctahedron* in the space orthogonal to the unit quaternion. Multiplication of these quaternions by the elements of the binary tetrahedral group T in (5) would give 24 sets of vertices of cuboctahedra $TQ = QT$ and leading to a total of 96 vertices. Repetition of some of the vertices under multiplication must be noted. Application of the group elements of $W(SO(7))_L$ on the highest weight $\Lambda^{(0)} = 2 + e_1 + e_2$ would lead to the set of quaternions

$$\begin{aligned} Q_L \ = \ &\{1 + 2e_1 \pm e_2,\ 1 + 2e_1 \pm e_3, \\ &\ 2 + e_1 \pm e_2,\ 2 + e_1 \pm e_3\} \quad (42) \end{aligned}$$

which represents a cube, although it is perhaps not quiet obvious to the reader. If one performs on (42) the Z_2 transformation of (17) or (18), one would get the set of quaternions

$$Q_R = \frac{1}{\sqrt{2}}(3 \pm e_1 \pm e_2 \pm e_3). \qquad (43)$$

This obviously represents the vertices of a cube in a space generated by the unit quaternions e_1, e_2, e_3. The quaternions of (43) would be obtained if the group $W(SO(7))_R$ is applied on the highest weight (0100). Multiplication of the set of quaternions of (42) by the elements of T' would not only give us the 96 elements of the orbit-(0010), but also classify vertices of the 24 sets of cubes. Therefore, the orbit-(0010) consists of 48 cells, made by 24 cubes and 24 cuboctahedra. Number of edges 288 can be determined by counting the number of nearest points to a given vertex and the number of faces 240 can be easily determined from the Euler formula.

The orbit-(0011) (truncated 24-cell)
We do not have prismatic cells in this polytope. This orbit has the same structure as the orbit-(1100) obtained by Dynkin diagram symmetry. It has 192 vertices, which decomposes under $W(SO(9))$ as an orbit of the size 192. The highest weight representing the orbit-(0011) is given by $\Lambda^{(0)} = 3+2e_1+e_2$. The application of $W(SO(7))_R$ on this weight would fix 3 and permutes the quaternions e_1, e_2, e_3 with a change of sign as well. The result is

$$3 \pm 2e_1 \pm e_2,\ 3 \pm 2e_2 \pm e_3,\ 3 \pm 2e_3 \pm e_1,\\ 3 \pm e_1 \pm 2e_2,\ 3 \pm e_2 \pm 2e_3,\ 3 \pm e_3 \pm 2e_1 \quad (44)$$

representing the vertices of a *truncated octahedron*. As we have discussed above, the multiplication of (44) by T would lead to the 192 vertices of the polytope and the classification of the vertices of cuboctahedra invariant under the 24 conjugate groups $W(SO(7))_R$. Action of the group $W(SO(7))_L$ on the same highest weight would lead to the quaternions

$$2+3e_1 \pm e_2,\ 2+3e_1 \pm e_3,\\ 3+2e_1 \pm e_2, 3+2e_1 \pm e_3 \quad (45)$$

representing a cube, although this is not so obvious because these vertices are in the space orthogonal to the unit vector $\frac{1}{\sqrt{2}}(1+e_1)$. Multiplying the set of quaternions of (45) by T' would lead to the vertices of 24 cubes in the orbit-(0011). If we perform the Dynkin diagram symmetry on (45), we would get the vertices of a cube in the orbit-(1100)

$$\frac{1}{\sqrt{2}}(5 \pm e_1 \pm e_2 \pm e_3). \quad (46)$$

In this orbit, we have 48 cells, 24 of each type, cubes and truncated octahedra. In addition, it has 240 faces and 384 edges.

The orbit-(0101) (cantellated 24-cell)
This orbit of size 288 decomposes under $W(SO(9))$ as $288 = 96 + 192$. The highest weight here is $\Lambda^{(0)} = (1+\frac{3}{\sqrt{2}}) + (1+\frac{1}{\sqrt{2}})e_1 + \frac{1}{\sqrt{2}}e_2 + \frac{1}{\sqrt{2}}e_3$. The action of $W(SO(7))_R$ on this weight will fix the scalar part of the quaternion, change the signs of e_1, e_2, e_3 and then permute. Then the quaternions

$$(1+\frac{3}{\sqrt{2}}) \pm (1+\frac{1}{\sqrt{2}})e_1 \pm \frac{1}{\sqrt{2}}e_2 \pm \frac{1}{\sqrt{2}}e_3\\ (1+\frac{3}{\sqrt{2}}) \pm \frac{1}{\sqrt{2}}e_1 \pm (1+\frac{1}{\sqrt{2}})e_2 \pm \frac{1}{\sqrt{2}}e_3 \quad (47)\\ (1+\frac{3}{\sqrt{2}}) \pm \frac{1}{\sqrt{2}}e_1 \pm \frac{1}{\sqrt{2}}e_2 \pm (1+\frac{1}{\sqrt{2}})e_3$$

represent the 24 vertices of a *small rhombicuboctahedron*. Application of the group $W(SO(7))_L$ on the highest weight would yield 12 quaternions

$$\{(2+\sqrt{2}) \pm e_1 \pm e_2,\ (2+\sqrt{2}) \pm e_2 \pm e_3\\ (2+\sqrt{2}) \pm e_3 \pm e_1\} \quad (48)$$

representing a *cuboctahedron*. Multiplying, either the quaternions in (48) by the set T' or the quaternions in (47) by the set T, will produce 288 quaternionic vertices of the polytope (0101) in the form of 24 sets of quaternions representing the vertices of the cells *small rhombicuboctahedra* and *cuboctahedra*. This orbit has 96 prismatic cells under the group $(D_3 \times Z_2)_L$. The orbit consists of 144 cells, 720 faces, 864 edges and 288 vertices.

The orbit-(0110) (bitruncated 24-cell)
This orbit is symmetric under the Dynkin diagram symmetry and has a larger symmetry $Aut(F_4)$. Its highest weight in terms of quaternions is

$$\Lambda^{(0)} = (2+\frac{3}{\sqrt{2}}) + (1+\frac{1}{\sqrt{2}})e_1 + (1+\frac{1}{\sqrt{2}})e_2 + \frac{1}{\sqrt{2}}e_3 \quad (49)$$

and has 288 vertices which under $W(SO(9))$ decomposes as $288 = 96 + 192$. The left and right copies of $W(SO(7))$ will lead to the same type of cell, though one of them is rotated with respect to the other by the Z_2 group element in (18). The application of $W(SO(7))_R$ on the highest weight in (49) will lead to the quaternionic set

$$(2+\frac{3}{\sqrt{2}}) \pm (1+\frac{1}{\sqrt{2}})e_1 \pm (1+\frac{1}{\sqrt{2}})e_2 \pm \frac{1}{\sqrt{2}}e_3,$$
$$(2+\frac{3}{\sqrt{2}}) \pm (1+\frac{1}{\sqrt{2}})e_1 \pm \frac{1}{\sqrt{2}}e_2 \pm (1+\frac{1}{\sqrt{2}})e_3$$
and
$$(2+\frac{3}{\sqrt{2}}) \pm \frac{1}{\sqrt{2}}e_1 \pm (1+\frac{1}{\sqrt{2}})e_2 \pm (1+\frac{1}{\sqrt{2}})e_3 \quad (50)$$

representing the vertices of a *truncated cube*. Application of Z_2 on (50) will result in the vertices of the *truncated cube* of the group $W(SO(7))_L$. Multiplication of the quaternions in (50) by the set T will classify the vertices of the 24 truncated cubes of $W(SO(7))_R$ and produce the 288 vertices of the polytope. It has altogether 48 cells, 24 of which is fixed by the conjugates of the group $W(SO(7))_L$. The orbit has 48 cells, 336 faces 576 edges and 288 vertices. This polytope does not possess cells of triangular prisms which follows from the discussions in Section 3.

The orbit-(1001) (runcinated 24-cell)
The polytope has the larger symmetry $Aut(F_4)$ since it is symmetric under the Dynkin diagram symmetry. The highest weight here is $\Lambda^{(0)} = (1+\sqrt{2})+e_1$ which leads to an orbit of size 144. Its decomposition under $W(SO(9))$ is $144 = 96 + 48$. The action of $W(SO(7))_R$ will result in the set of quaternions

$$(1+\sqrt{2}) \pm e_1,\ (1+\sqrt{2}) \pm e_2,\ (1+\sqrt{2}) \pm e_3 \quad (51)$$

which represents the vertices of an octahedron. The vertices of the octahedron invariant under the group $W(SO(7))_L$ are obtained by the action of Z_2 group element in (18). 24 octahedra of $W(SO(7))_R$ are obtained by multiplying the set of quaternions in (51) by the elements of the binary tetrahedral group T, which also produces the 144 quaternionic vertices of the polytope. The same set of 144 quaternions can also be classified as 24 set of octahedral vertices obtained from (51) by the action of the Z_2 group element and then multiplying them by the set T' of quaternions. The discussion in Section 3 indicates that this orbit has prismatic cells. The action of the group elements of $(D_3 \times Z_2)_R$ in (33) produces the six quaternions

$$(\sqrt{2}+1)e_1,\ (\sqrt{2}+1)e_3,$$
$$(\sqrt{2}+1)e_2,\ (\sqrt{2}+1)\omega_0+\omega_1,$$
$$(\sqrt{2}+1)\omega_0+\omega_3,\ (\sqrt{2}+1)\omega_0+\omega_2 \quad (52)$$

representing the vertices of a triangular prism. We obtained 96 sets of prisms with the techniques discussed above. Since the symmetry group of the polytope is $Aut(F_4)$ one obtains the prisms fixed by the conjugates of the $(D_3 \times Z_2)_L$ by employing Z_2. Altogether we have $24 + 24 + 96 + 96 = 240$ cells of the orbit-(1001). The polytope has then 144 vertices, 240 cells, 672 faces and 576 edges.

The orbit-(1011) (runcitruncated 24-cell)
The highest weight is the quaternion $\Lambda^{(0)} = (3 + \sqrt{2}) + 2e_1 + e_2$. It has 576 vertices and decomposes under $W(SO(9))$ as $576 = 192 + 384$. The group $W(SO(7))_R$, fixing the scalar part of the quaternion, generates the set of quaternions given by

$$(3+\sqrt{2}) \pm 2e_1 \pm e_2,\ (3+\sqrt{2}) \pm 2e_2 \pm e_3,$$
$$(3+\sqrt{2}) \pm 2e_3 \pm e_1, (3+\sqrt{2}) \pm e_1 \pm 2e_2,$$
$$(3+\sqrt{2}) \pm e_2 \pm 2e_3,\ (3+\sqrt{2}) \pm e_3 \pm 2e_1, \quad (53)$$

which represents the 24 vertices of a *truncated octahedron*. Multiplication of these quaternions by the set T will create $24 \times 24 = 576$ vertices of the polytope (1011). Applying $W(SO(7))_L$ on the highest weight of the orbit-(1011) would yield the set of quaternions obtained from the following set

$$(1+\frac{5}{\sqrt{2}}) \pm (1+\frac{1}{\sqrt{2}})e_1 \pm \frac{1}{\sqrt{2}}e_2 \pm \frac{1}{\sqrt{2}}e_3,$$
$$(1+\frac{5}{\sqrt{2}}) \pm \frac{1}{\sqrt{2}}e_1 \pm (1+\frac{1}{\sqrt{2}})e_2 \pm \frac{1}{\sqrt{2}}e_3, \quad (54)$$
$$(1+\frac{5}{\sqrt{2}}) \pm \frac{1}{\sqrt{2}}e_1 \pm \frac{1}{\sqrt{2}}e_2 \pm (1+\frac{1}{\sqrt{2}})e_3$$

using $Z_2 = [\frac{1}{\sqrt{2}}(e_2+e_3), -e_2]$ transformation. Of course, the set of quaternions in (54) represent a *small rhombicuboctahedron*. However, one should keep in mind that the quaternions in (54) are not the elements of the orbit-(1011) but belong to the orbit-(1101) as a cell of $W(SO(7))_R$. The reason we displayed those quaternions in (54) is because the vertices of *small rhombicuboctahedron* are best known in this form. As we discussed in Section 3 this orbit has a hexagonal prism under the group $(D_3 \times Z_2)_R$ and triangular prism under the group $(D_3 \times Z_2)_L$. Therefore, it consists of 24 *truncated octahedra*, 24 *small rhombicuboctahedra*, 96 hexagonal prisms and 96 triangular prisms. The orbit has 240 cells, 576 vertices, 1104 faces and 1440 edges.

The orbit-(0111) (cantitruncated 24-cell)
The highest weight here is the quaternion

$$\Lambda^{(0)} = (3+\frac{3}{\sqrt{2}})+(2+\frac{1}{\sqrt{2}})e_1+(1+\frac{1}{\sqrt{2}})e_2+\frac{1}{\sqrt{2}}e_3, \quad (55)$$

which leads to the 576 vertices decomposing under $W(SO(9))$ as $576 = 192 + 384$. When the group $W(SO(7))_R$ acts on this quaternion it will produce 48 quaternions representing the vertices of a *great rhombicuboctahedron*. Similarly, the action of the group $W(SO(7))_L$ would produce the vertices of a *truncated cube*. Either multiplying the quaternionic vertices of the *great rhombicuboctahedron* by the set T or the quaternionic vertices of the *truncated cube* by the set T' will lead to 576 quaternionic vertices of the polytope (0111). The vertices of its Dynkin symmetric polytope (1110) will be obtained from the vertices of (0111) by applying the group element Z_2 in (18). This polytope has a triangular prism under the symmetry $(D_3 \times Z_2)_L$. The polytope overall has 144 cells, 576 vertices, 720 faces and 1152 edges.

The orbit-(1111) (omnitruncated 24-cell)
This is the largest $W(F_4)$ orbit with a size of 1152, which is the order of the Coxeter-Weyl group $W(F_4)$ and naturally decomposes under $W(SO(9))$ as $1152 = 384 + 384 + 384$. It is the last member of the semi-regular polytopes of the group $W(F_4)$. It is represented with the highest weight

$$\Lambda^{(0)} = (3+\frac{5}{\sqrt{2}})+(2+\frac{1}{\sqrt{2}})e_1+(1+\frac{1}{\sqrt{2}})e_2+\frac{1}{\sqrt{2}}e_3. \quad (56)$$

The action of the left and right groups will produce the same type of cells. The action of group $W(SO(7))_R$ would produce the vertices of a *great rhombicuboctahedron*. This is obvious because (56) differs from (55) only by the scalar part of the quaternion left invariant by the group $W(SO(7))_R$. The action of group $W(SO(7))_L$ will produce 48 set of quaternions obtained from (56) by a Z_2 rotation. A multiplication of the 48 quaternions obtained by the group $W(SO(7))_R$ by the 24 quaternions of the binary tetrahedral group T will produce exactly $48 \times 24 = 1152$ quaternionic vertices of the polytope (1111). From the discussions in Section 3, it follows that the orbit has hexagonal prisms of two types that are Z_2 symmetric with respect to each other. The polytope has the symmetry $Aut(F_4)$ of order 2304 having 1152 vertices,240 cells, 1392 faces and 2304 edges.

5. Conclusion

This paper dealt with the regular and semi-regular polytopes of the Coxeter-Weyl group $W(F_4)$ and its extension $Aut(F_4)$ by Dynkin diagram symmetry. The classifications of the polytopes here can be found in reference [10], but were obtained without the technique discussed here. Determination of the vertices of the polytopes with the highest weight technique of Lie algebra and the symmetry groups represented by quaternions are the two novel aspects of our technique. The cells of a given polytope are classified with their symmetry groups that are the 3D-subgroups of the Coxeter-Weyl group $W(F_4)$. As a reference we give the list of the platonic and Archimedean solids transforming under the Coxeter-Weyl groups $W(SO(7))$ and $W(SU(4))$ in Table II. We present the polytopes in Table III with the relevant cell symmetries as a summary of what has been discussed in the paper.

TABLE II: Platonic and Archimedean solids designated by Dynkin labels of $W(SO(7))$ and $W(SU(4))$ and prismatic polyhedra [5].

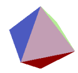
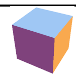
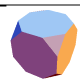
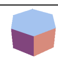
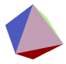
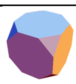
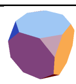
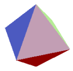
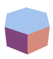
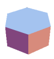
Symmetry	Order	Dynkin label	Name	Figure
$W(SU(4))$	24	(100) or (001)	Tetrahedron	
$W(SU(4))$	24	(110) or (011)	Truncated tetrahedron	
$W(SO(7))$	48	(100)	Octahedron	
$W(SO(7))$	48	(001)	Cube	
$W(SO(7))$	48	(010)	Cuboctahedron	
$W(SO(7))$	48	(110)	Truncated octahedron	
$W(SO(7))$	48	(011)	Truncated cube	
$W(SO(7))$	48	(101)	Small rhombicuboctahedron	
$W(SO(7))$	48	(111)	Great rhombicuboctahedron	
$D_3 \times Z_2$	12		Triangular prism	
$D_3 \times Z_2$	12		Hexagonal prism	
$D_4 \times Z_2$	16		Octagonal prism	

TABLE III: $W(F_4)$ orbits as regular and semi-regular polytopes.

	Cell counts by symmetry				Element counts			
Dynkin label	$W(SO(7))_L$ cells (24)	$(D_3 \times Z_2)_L$ cells (96)	$(D_3 \times Z_2)_R$ cells (96)	$W(SO(7))_R$ cells (24)	Cells	Faces	Edges	Vertices
(0001)					24	96	96	24
(0010)					48	240	288	96
(0011)					48	240	384	192
(0101)					144	720	864	288
*(0110)					48	336	576	288
*(1001)					240	672	576	144
(1011)					240	1104	1440	576
(0111)					144	720	1152	576
*(1111)					240	1392	2304	1152

* This orbit has the symmetry $Aut(F_4) \approx W(F_4) : Z_2$ of order 2304.

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