The Exact Solution of Fokker-Planck Equation for Brownian Motion

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In this article, we are interested in deriving the Fokker-Planck equation, which is based on the Langevin equation for Brownian motion and then find an explicit form of a certain probability distribution by using a mathematical method. We also calculated the velocity moments for this system.

1. Introduction

Theoretical physics can be roughly viewed as the study of solutions of differential equations and the modeling of natural phenomena by deterministic solutions of differential equations.

The quest for a mathematical description of the Brownian trajectories led to a new class of differential equations, namely the so-called stochastic differential equations. Such equations can be regarded as the generalization pioneered by Paul Langevin of Newtonian mechanical equations that are driven by independent stochastic increments obeying either a Gaussian (white Gaussian noise) or the Poisson statistics (white Poisson noise). This yields a formulation of the Fokker-Planck equation (master equation) in terms of a nonlinear Langevin equation generally driven by multiplicative white Gaussian (Poisson) noise(s) [1].

As the aforementioned independent increments correspond to no bounded trajectory variations, the integration of such differential equations must be given a more general meaning. This led to the stochastic integration calculus of either the Ito type or the Stratonovich type. In recent years, this method of modeling the statistical mechanics of generally nonlinear systems driven by random forces has been developed further to account for physically more realistic noise sources possessing a finite or even infinite noise-correlation time.

The general Fokker-Planck equation for one variable *x* has the form [2]:

$$\frac{\partial w}{\partial t}=\left[-\frac{\partial}{\partial x}D^{(1)}(x)+\frac{\partial^2}{\partial x^2}D^{(2)}(x)\right] \qquad (1)$$

In this equation, $D^{(1)}(x)$ is called the drift coefficient and $D^{(2)}(x)>0$ is the diffusion coefficient. Eqn. (1) is the equation of motion for the distribution function $w(x,t)$.

2. Langevin Equation for Brownian Motion

If a small particle of mass immersed in a fluid, a friction force will act on the particle. The simplest expression for such a friction force is given by Stokes law [3]:

$$F_c=-\alpha v \qquad (2)$$

Therefore, the equation of motion for the particle in the absence of additional forces reads:

$$m\dot{v}+\alpha v=0 \qquad (3)$$

$$\dot{v}+\gamma v=0,\gamma=\frac{\alpha}{m}=\frac{1}{\tau} \qquad (4)$$

Thus an initial velocity $v(0)$ decreases to zero with the relaxation time $\tau=\frac{1}{\gamma}$ according to the formula

$$v(t)=v(0)e^{\frac{-t}{\tau}} \qquad (5)$$

The physics behind the friction is that the molecules of the fluid collide with the particle. The momentum of the particle is transferred to the molecules of the fluid and the velocity of the particle therefore decreases to zero. If the mass of the small particle is still large compared to the mass of the molecules, one expects Eqn. (3) to be valid approximately. Eqn. (3) must be modified so that it leads to correct thermal energy. The modification consists in adding a fluctuation force $F_f(t)$ on the right-hand side of Eqn. (3).

Then, the total force of the molecules acting on the small particle is decomposed into a continuous damping force $F_c(t)$ and a fluctuating force, $F_f(t)$, satisfying

$$F(t) = F_c(t) + F_f(t) = -\alpha v(t) + F_f(t) \quad (6)$$

By inserting Eqn. (6) into Eqn. (3) and dividing by the mass, we get the equation of motion:

$$\dot{v} + \gamma v = \Gamma(t) \quad (7)$$

Here, we have introduced the fluctuating force per unit mass as

$$\Gamma(t) = \frac{F_f(t)}{m} \quad (8)$$

which is called the Langevin equation for Brownian motion.

3. How to Derive Fokker-Planck Equation in Brownian Motion by Langevin Equation

We start with Eqn. (7), where v is velocity of particle and defines $\Gamma(t)$.

The white noise, on the other hand, is a Fourier transformation having the form [4]:

$$w(v,t) = \int_{-\infty}^{+\infty} e^{i\lambda z} Z(\lambda,t) d\lambda \quad (9)$$

$$Z(\lambda,t) = \int_{-\infty}^{+\infty} e^{-i\lambda v} w(v,t) dv \quad (10)$$

$$Z(\lambda,t) = \left\langle e^{-i\lambda v} \right\rangle \quad (11)$$

$$Z_t = -i\lambda \left\langle v_t e^{-i\lambda v} \right\rangle = -i\lambda \left\langle (-\gamma v + \Gamma(t)) e^{-i\lambda v} \right\rangle \quad (12)$$

$$= i\lambda\gamma \left\langle v e^{-i\lambda v} \right\rangle - i\lambda \left\langle \Gamma(t) e^{-i\lambda v} \right\rangle \quad (13)$$

Since the force $\Gamma(x,t)$ is Gaussian, we can use the standard trick of the theory of Langevin equation, it is called the Novikov theory [5,6]:

$$\left\langle \Gamma(x,t) \exp \sum i\lambda_j v(x_j,t) \right\rangle = -i \sum \mathrm{K}(x - x_j) \lambda_j Z \quad (14)$$

$$z_t = -\gamma \lambda Z_\lambda - \lambda^2 Z \quad (15)$$

Now, we multiply $e^{i\lambda v}$ by Z_t and making use of Eqns. (9) and (10), we get

$$\int \frac{\partial z}{\partial t} e^{i\lambda v} dv = -\gamma \int \lambda e^{i\lambda v} \frac{\partial z}{\partial \lambda} - \int \lambda^2 e^{i\lambda v} d\lambda \quad (16)$$

$$= i\gamma \frac{\partial}{\partial v}\left(-iv \int e^{i\lambda v} Z d\lambda \right) + \frac{\partial^2 w(v,t)}{\partial v^2} \quad (17)$$

$$\frac{\partial w(v,t)}{\partial t} = \frac{\partial}{\partial v}(\gamma v w(v,t)) + \frac{\partial^2 w(v,t)}{\partial v^2} \quad (18)$$

Eqn. (18) is called the Fokker-Planck equation for Brownian motion. Now, we are interested in calculating the distribution function.

4. Exact Solution of Fokker-Planck Equation

Here, we will consider $D^{(1)}$ and $D^{(2)}$, which are t-independent, and $D^{(1)}$ is linear in v and $D^{(2)}$ is constant.

It follows that

$$D^{(1)} = -\gamma v \quad , \quad D^{(2)} = Cte \quad (19)$$

The equation for distribution function now reads as

$$\frac{\partial w(v,t)}{\partial t} = \gamma \frac{\partial}{\partial v}\left(v w(v,t) \right) + Cte \frac{\partial^2 w(v,t)}{\partial v^2} \quad (20)$$

With the initial condition

$$w(v, t=0) = \delta(v) \quad (21)$$

The solution of Eqn. (20) is best found by making a Fourier transformation in v, i.e.,

$$w(v,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda z} Z(\lambda,t) d\lambda \quad (22)$$

We consider to initial condition (Eqn. (21)), where the initial condition for the Fourier transformation is [7] given as

$$Z(\lambda,t) = e^{-i\lambda t} \quad (23)$$

The first order of Eqn. (15) may be solved by the methods of characteristics [8]. The solution of Eqn. (15) reads as

$$Z(\lambda,t) = \exp\left[-\frac{c\lambda^2 (1 - e^{-2\gamma t})}{2\gamma} \right] \quad (24)$$

By performing the integral in Eqn. (22), we finally get the Gaussian distribution

$$w(v,t)=\sqrt{\frac{\gamma}{2\pi c(1-e^{-2\gamma t})}}\exp\left[\frac{-\gamma v^2}{2c(1-e^{-2\gamma t})}\right] \quad (25)$$

Eqn. (25) is valid for both positive and negative γ.

5. Calculation of n^th Moment for Stationary State

To obtain the n^th moment we have [9]

$$M_n=\langle v^n\rangle=\int v^n w(v,t)dv \quad (26)$$

$$w_{st}=\sqrt{\frac{\gamma}{2\pi c}}\exp\left[-\frac{\gamma v^2}{2c}\right] \quad (27)$$

Now, for this system we can write

$$M_1=\langle v\rangle=\sqrt{\frac{\gamma}{2\pi c}}\int v e^{-\frac{\gamma v^2}{2c}}dv=0 \quad (28)$$

$$M_2=\langle v^2\rangle=\sqrt{\frac{\gamma}{2\pi c}}\int v^2 e^{-\frac{\gamma v^2}{2c}}dv=\sqrt{\frac{\gamma}{2\pi c}}\frac{\sqrt{2\pi}}{\gamma^{\frac{3}{2}}} \quad (29)$$

$$M_3=\langle v^3\rangle=\sqrt{\frac{\gamma}{2\pi c}}\int v^3 e^{-\frac{\gamma v^2}{2c}}dv=0 \quad (30)$$

$$M_4=\langle v^4\rangle=\sqrt{\frac{\gamma}{2\pi c}}\int v^4 e^{-\frac{\gamma v^2}{2c}}dv=\sqrt{\frac{\gamma}{2\pi c}}\frac{3\sqrt{2\pi}}{\gamma^{\frac{5}{2}}} \quad (31)$$

$$M_5=\langle v^5\rangle=0 \quad (32)$$

$$M_6=\langle v^6\rangle=\sqrt{\frac{\gamma}{2\pi c}}\int v^6 e^{-\frac{\gamma v^2}{2c}}dv=\sqrt{\frac{\gamma}{2\pi c}}\frac{15\sqrt{2\pi}}{\gamma^{\frac{7}{2}}} \quad (33)$$

Finally, we can calculate the average of kinetic energy for this system in stationary state, which is:

$$\langle E\rangle=\frac{1}{2m}\langle v^2\rangle=\sqrt{\frac{m^2}{4c\gamma^2}} \quad (34)$$

6. Conclusion

In this article, the distribution function is exactly calculated in terms of velocity and time for Brownian motion by using the known distribution. Then, with the help of the distribution function in stationary state, we could calculate the average value of the kinetic energy. We perceive that the odd order of the moment is equal to zero and the even order of the moment is limited. In general, the Brownian motion has indeed many more applications that one expects. Many common occurrences which we come across have somehow been linked to the characteristics of Brownian motion. In particular, the fractal theory and the theory of continuum walks are of great significance. Brownian noise has long been recognized as a form of unavoidable interference in the transmission, while the estimation of floods is a rather recent discovery that is applicable to the environment. In this article, various applications of the Brownian motion have been mentioned. It is obvious that the Brownian motion is not just a physical theory that is solely applied to the world of science and technology. Instead, it covers quite a number of interesting aspects of life without our being aware of its role. Brownian model should also be very useful for the uncertainly theory which has been developing extensively in recent times.

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