General Relativistic Theory of Oblate Spheroidal Gravitational Field

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In this article, present an application of General Relativity theory to the oblate spheroidal space time. A metric tensor for this gravitational field is constructed along with field equations and various solutions are suggested. Also, the motion of test particles and photons in this field are investigated. The gravitational field Lagrangian is constructed and then used to derive the expression for energy and angular momentum conservation. This expression for the conservation of angular momentum does not depend on gravitational potential. It is of the same form as in the Schwarzschild space time and Newton's dynamical theory of gravitation. The effect of the oblate spheroidal nature of the Sun and planets on some gravitational phenomena is also investigated. The metric tensor is used to predict theoretical values for the Pound Rebka experiments if it were performed on other planets. The shift in frequency is obtained as $z \cong 2.578 \text{ x } 10^{-15}$ for planet Earth. This value is quite close to that obtained by Pound and Rebka $(z \cong 2.45 \text{ x } 10^{-15})$.

1. Introduction

Recently [1-5], we introduced an approach to obtain the metric tensor exterior to a static oblate spheroidal mass. Here, we present our major results and outline their physical significance. A covariant metric tensor having six non-zero components is constructed. This tensor is used to derive Einstein's gravitational field equations interior and exterior to the oblate spheroid. Solutions are also constructed for these field equations. To authenticate the validity of this approach, gravitational phenomena such as gravitational time dilation, gravitational length contraction and gravitational spectral shift are studied in the gravitational field exterior to astrophysical bodies in the Solar System.Basically, emphasis is made on gravitational sources with time independent and axially-symmetric distributions of mass within oblate spheroids, characterized by two typical integrals of geodesic motion, namely, energy and angular momentum. From an astrophysical point of view, although such an assumption is not necessary, it could prove useful because of its equivalence to the assumption that the gravitational source is changing slowly in time so that partial time derivatives are negligible as compared to the spatial ones. It is stressed that the mass source considered is not the most arbitrary one from a theoretical point of view. On the other hand, many astrophysically interesting systems are usually assumed to be time independent (or static from another point of view) and axially symmetric continuous sources [6].

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2. Metric Tensor Exterior to a Homogeneous Oblate Spheroid

This section outlines the construction of the covariant metric tensor exterior to a homogeneous oblate spheroid.

In order to obtain a metric tensor for this gravitational field, we let a spherically symmetric body ("Schwarzschild's body") to be transformed through deformation into an oblate spheroidal body in such a way that its density ρ_0 and total mass M remain the same and its surface parameter are given in oblate spheroidal coordinates [7] as

$$\xi = \xi_0; \quad \text{constant} \tag{1}$$

Then, we remark that the general relativistic field equations, exterior to an oblate spheroidal body, are mathematically equivalent to those of the spherical symmetric body. This is because, they are both tensorially the same. Hence, they are only related by the transformation from spherical to oblate spheroidal coordinates. Therefore, to get the corresponding invariant world line element in the exterior region of an oblate spheroidal mass, the function, $f(r)$, (an arbitrary function in the spherically symmetric field) is replaced by the corresponding function $f(\eta, \xi)$ exterior to a homogeneous oblate spheroidal body. A sound and astrophysically satisfactory approximate expression for the function $f(\eta, \xi)$ is obtained by equating it to the gravitational scalar potential exterior to the distribution of mass within oblate spheroidal regions [8]. Next, coordinates are transformed from spherical to oblate spheroidal as

$$(ct, r, \theta, \phi) \rightarrow (ct, \eta, \xi, \phi) \tag{2}$$

on the right-hand side of the line element. A simplification yields the following components of the metric tensor in the region exterior to a homogeneous oblate spheroid in oblate spheroidal coordinates [2]

$$g_{00} = \left(1 + \frac{2}{c^2} f(\eta, \xi)\right) \tag{3}$$

$$g_{11} = -\frac{a^2}{1+\xi^2-\eta^2}\left[\eta^2\left(1+\frac{2}{c^2} f(\eta,\xi)\right)^{-1} + \frac{\xi^2(1+\xi^2)}{(1-\eta^2)}\right] \tag{4}$$

$$g_{12} = g_{21} = -\frac{a^2\eta\xi}{1+\xi^2-\eta^2}\left[1-\left(1+\frac{2}{c^2} f(\eta,\xi)\right)^{-1}\right] \tag{5}$$

$$g_{22} = -\frac{a^2}{1+\xi^2-\eta^2}\left[\xi^2\left(1+\frac{2}{c^2} f(\eta,\xi)\right)^{-1} + \frac{\eta^2(1-\eta^2)}{(1+\xi^2)}\right] \tag{6}$$

$$g_{33} = -a^2(1+\xi^2)(1-\eta^2) \tag{7}$$

$$g_{\mu\nu} = 0; \quad \text{otherwise} \tag{8}$$

This metric has been very instrumental in our development of general relativistic mechanics in gravitational fields exterior to homogeneous oblate spheroids. The covariant metric tensor obtained for gravitational fields exterior to oblate spheroidal masses has two additional non-zero components, g_{12} and g_{21}, which are not found in Schwarzschild field.

Thus, the extension from Schwarzschild field to homogeneous oblate spheroidal gravitational fields has produced two additional non zero tensor components and thus this metric tensor field is unique. This confirms the assertion that oblate spheroidal gravitational fields are more complex than spherical fields and consequently general relativistic mechanics in this field is more involved.

3. Field Equations Exterior to an Oblate Spheroid

Here, the metric tensor is used to derive gravitational field equations exterior to a homogeneous oblate spheroidal mass.

After the construction of the covariant metric tensor, the following standard method is used to derive field equations exterior to a homogeneous oblate spheroid. To obtain the contravariant metric tensor for the gravitational field exterior to an oblate spheroid, $g^{\mu\nu}$, we use the fact that $g^{\mu\nu}$ is the cofactor of $g_{\mu\nu}$ in g divided by g [9]. That is

$$g^{\mu\nu} = \frac{\text{cofactor of } g_{\mu\nu} \text{ in g}}{\text{g}} \tag{9}$$

Where,

$$\text{g} = \begin{vmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{vmatrix} \tag{10}$$

The coefficients of affine connection $\Gamma^{\sigma}_{\mu o}$ for any gravitational field are defined in terms of the covariant and contravariant metric tensor of space-time as [9];

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\nu}\left(g_{\mu\nu,\lambda} + g_{\nu\lambda,\mu} - g_{\mu\lambda,\nu}\right) \tag{11}$$

Where, the comma denotes partial differentiation with respect to λ, μ and ν. We have constructed the known 64 coefficients of affine connection for this gravitational field [10]. The curvature tensor or the Riemann-Christoffel tensor $R^{\delta}_{\alpha\beta\sigma}$ for this field is defined in terms of the coefficients of affine connection [11] as

$$R^{\delta}_{\alpha\beta\sigma} = \Gamma^{\delta}_{\alpha\sigma,\beta} - \Gamma^{\delta}_{\alpha\beta,\sigma} + \Gamma^{\varepsilon}_{\alpha\sigma}\Gamma^{\delta}_{\varepsilon\beta} - \Gamma^{\varepsilon}_{\alpha\beta}\Gamma^{\delta}_{\varepsilon\sigma} \tag{12}$$

Where, the comma denotes partial differentiation with respect to β and σ. We have also constructed the 256 components of this tensor for homogeneous oblate spheroidal gravitational fields. From the curvature tensor $R^{\delta}_{\alpha\beta\sigma}$ for this gravitational field, we have defined a second rank tensor $R_{\alpha\beta}$ (called the Ricci tensor) for the gravitational field exterior to the oblate spheroid [11] as

$$R_{\alpha\beta} = R^{\delta}_{\alpha\beta\delta} \tag{13}$$

The 16 components of this tensor for the static homogeneous oblate spheroids have been constructed. From the Ricci tensor for our gravitational field, we deduced a scalar R defined by

$$R = R_{\alpha}^{\alpha} = g^{\alpha\beta} R_{\alpha\beta} \tag{14}$$

This is called the curvature scalar for homogeneous spheroidal fields.

It is well known that for a region exterior to any astrophysical body, the general relativistic field equations are given tensorially as [12]

$$G_{\mu\nu} = 0 \tag{15}$$

Where, $G_{\mu\nu}$ is the Einstein tensor and given explicitly as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{16}$$

Where, $R_{\mu\nu}$ is the Ricci tensor, R the curvature Scalar and $g_{\mu\nu}$ is the covariant metric tensor for the field. The Einstein field equations for the gravitational field exterior to homogeneous oblate spheroids are then built up. The field equations obtained [10] can be written more explicitly in terms of the metric tensor and affine connections (see Appendix). The derived gravitational field equations are second order partial differential equations that can be solved and interpreted. All its mathematically possible solutions may then be distinguished by physical considerations, such as consistency with astrophysical or astronomical observations, the data and facts. Hence, in principle, our arbitrary function, $f(\eta,\xi)$, which uniquely and completely determines the solution of Einstein's gravitational metric tensor field exterior to the static homogeneous oblate spheroidal mass or pressure distributions can be found.

4. Solutions to Gravitational Field Equations Exterior to Oblate Spheroid

In this section, solutions to the field equations exterior to homogeneous oblate spheroidal masses are constructed.

By assuming that any two of the field equations possess a common simultaneous solution of the function, $f(\eta,\xi)$, a number of mathematical contradictions arise. Therefore, it is concluded that in the space time exterior to a static homogeneous distribution of mass within a homogeneous oblate spheroidal region in the universe, no two field equations possess any common solution. Consequently, these field equations may only be solved separately and their different solutions applied whenever and wherever necessary and useful in physical theories.

We outline the solution for the third field equation, which can be written in terms of the metric tensor and Ricci tensor as

$$-g^{00}R_{00} - g^{11}R_{11} - g^{22}R_{22} + g^{33}R_{33} - 2g^{12}R_{12} = 0 \tag{17}$$

Writing various terms of Eqn. 17 explicitly in terms of the metric tensor only, we obtain our explicit field equation. Now, we realize that our covariant metric tensor, Eqns. 3-8 can be written as

$$g_{\mu\nu}(\eta,\xi) = h_{\mu\nu}(\eta,\xi) + f_{\mu\nu}(\eta,\xi) \tag{18}$$

Where, $h_{\mu\nu}(\eta,\xi)$ are the well-known pure empty space components and $f_{\mu\nu}(\eta,\xi)$ are the contributions due to the oblate spheroidal mass distribution. Consequently, as the mass distribution decays out, i.e., $f_{\mu\nu}(\eta,\xi) \to 0$, it leads to $g_{\mu\nu}(\eta,\xi) \to h_{\mu\nu}(\eta,\xi)$. That is, the metric tensor reduces to the pure empty space metric tensor as the distribution of mass decays out.

Also,

$$g^{\mu\nu}(\eta,\xi) = h^{\mu\nu}(\eta,\xi) + f^{\mu\nu}(\eta,\xi) \tag{19}$$

Where, $h^{\mu\nu}(\eta,\xi)$ are the well-known pure empty space components and $f^{\mu\nu}(\eta,\xi)$ are the contributions due to the oblate spheroidal mass distribution [10].

To begin the explicit formulation of the R_{33} field equation, first of all we note that all terms of the order of c^0 cancel out identically since the empty space time metric tensor $h_{\mu\nu}$ independently satisfies the homogeneous R_{33} field equation. Therefore, the lowest order term we expect in the exterior R_{33} field equation is c^{-2} term. Hence, in order to formulate the exterior R_{33} field equation of order, c^{-2}, we decompose our covariant metric tensor $g_{\mu\nu}$ into pure empty space part $h_{\mu\nu}$ (of order c^0 only) and the nonempty space part $f_{\mu\nu}$ of order c^{-2} or higher). Similarly, let the contra-variant metric tensor $g^{\mu\nu}$ be decomposed into pure empty space part $h^{\mu\nu}$ (of order c^0 only) and the nonempty space part $f^{\mu\nu}$ (of order c^{-2} or higher).

Substituting explicit expressions for Eqns.18 and 19 from [10] into Eqn. 17 and neglecting all terms of order c^0, the exterior R_{33} field equation can be written as;

$$K_1(\eta,\xi) f_{\eta\eta} + K_2(\eta,\xi) f_{\eta\xi} + K_3(\eta,\xi) f_{\xi\xi} +$$

$$K_4(\eta,\xi) f_{\eta} + K_5(\eta,\xi) f_{\xi} + K_6(\eta,\xi) f = 0 \quad (20)$$

Where, $K_i(\eta,\xi)$, $i = 1\ldots6$ are functions of (η,ξ).

In the exterior oblate spheroidal space time [8]

$$\xi \geq \xi_0 \ and \ -1 \leq \eta \leq 1; \xi_0 = constant \quad (21)$$

Let us now seek a solution for the R_{33} field (Eqn. 20) in the form of a power series as

$$f(\eta,\xi) = \sum_{n=0}^{\infty} P_n^+(\xi)\eta^{\eta} \quad (22)$$

Where, P_n^+ is a function to be determined for each value of n.

Substituting the proposed function into the field equation and taking into consideration the fact that $\{\eta^{\eta}\}_{n=0}^{\infty}$ is a linearly independent set, we can equate the coefficients of η^{η} on both sides of the obtained equation.

From the coefficients of η^0 of, we obtain the equation

$$\begin{aligned} 0 = {} & a^3\xi^3\left(1+\xi^2-a^2\xi^4\right)P_2^+(\xi) + 2a^3\xi^3(1+\xi^2)^2\left[P_1^+(\xi)\right]' + a^3\xi^2(1+\xi^2)P_1^+(\xi) \\ & + a^3\xi^3(1+\xi^2)^2\left[P_0^+(\xi)\right]'' + (1+\xi^2)\left(-1-2a^2\xi^2-\xi^2-a^2\xi^3+4a^2\xi^5\right)\left[P_0^+(\xi)\right]' \\ & + \left[2a^3\xi\left(4-2\xi^2-a^4\xi^4-a^4\xi^6\right)\right]P_0^+(\xi) \end{aligned} \quad (23)$$

This equation (Eqn. 23) is the first recurrence differential equation for unknown functions. All the other recurrence differential equations can follow from it yielding infinitely many recurrence differential equations. These can be used to determine all the unknown functions.

The following points can thus be made. Firstly, Eqn. 23 determines P_2^+ in terms of P_0^+ and P_1^+. Similarly, the other recurrence differential equations will determine the other unknown functions, $P_3^+ \ldots$, in terms of P_0^+ and P_1^+. Secondly, we note that we have the freedom to choose our arbitrary functions to satisfy the physical requirements or needs of any particular distribution or area of application.

Let us now recall that for any gravitational field [13],

$$g_{00} \cong 1 + \frac{2}{c^2}\Phi \quad (24)$$

Where, Φ is Newton's gravitational scalar potential for the field under consideration. Thus we can then deduce that the unknown function in the field equation can be given approximately as

$$f(\eta,\xi) \cong \Phi^+(\eta,\xi) \quad (25)$$

Where, $\Phi^+(\eta,\xi)$, is Newton's gravitational scalar potential exterior to a homogeneous oblate spheroidal mass. Recently [14], it has been shown that

$$\Phi^+(\eta,\xi) = B_0 Q_0(-i\xi)P_0(\eta) + B_2 Q_2(-i\xi)P_2(\eta) \quad (26)$$

Where, Q_0 and Q_2 are the Legendre functions linearly independent of Legendre polynomials P_0 and P_2, respectively, and B_0 and B_2 are constants.

Let us now seek our analytical exterior solution (Eqn. 22) to be as close as possible to the approximate exterior solution (Eqn. 23). Now, since the approximate solution does not possesses any term in the first power of η, let us choose

$$P_0^+(\xi) \equiv B_0 Q_0(-i\xi) + B_2 Q_2(-i\xi) \quad (27)$$

and

$$P_1^+(\xi) \equiv 0 \quad (28)$$

Hence, we can write $P_2^+(\xi)$ in terms of $P_0^+(\xi)$ as;

$$P_2^+(\xi) = -\frac{\left(1+\xi^2\right)^2}{\left(1+\xi^2-a^2\xi^4\right)}\left[P_0^+(\xi)\right]'' - \frac{2\left(1+\xi^2\right)\left(3a^2\xi^2+4a^2\xi^5-\xi^2-1\right)}{a^2\xi^3}\left[P_0^+(\xi)\right]'$$
$$-2\left[\frac{1-2a^3\xi^2-a^7\xi^4-a^7\xi^6+a^3}{a^3\xi^2\left(1+\xi^2-a^2\xi^4\right)}\right]P_0^+(\xi) \quad (29)$$

We now remark that the first three terms of our series solution converge everywhere in the exterior space time. We also remark that our solution of the order c^0 may be written as

$$f^+(\eta,\xi) = \Phi^+(\eta,\xi) + \Phi_0^+(\eta,\xi) \quad (30)$$

Where, $\Phi^+(\eta,\xi)$ is the corresponding Newtonian gravitational scalar potential and $\Phi_0^+(\eta,\xi)$ is the pure Einsteinian (general relativistic) or post Newtonian correction of the order c^0.

Hence, we deduce that our exterior analytical solution is of the general form

$$f(\eta,\xi) = \Phi^+(\eta,\xi) + \Phi_0^+(\eta,\xi) + \sum_{n=1}^{\infty}\Phi_{2n}^+(\eta,\xi) \quad (31)$$

Interestingly, the single dependent function f in our solution turns out to be the corresponding well known pure Newtonian exterior gravitational scalar potential augmented by hitherto unknown pure Einsteinian (or general relativistic or post-Newtonian) gravitational scalar potential terms of orders $c^0, c^{-2}, c^{-4}, \ldots$. Hence, this solution reveals a hitherto unknown sense in which the exterior Einstein's geometrical gravitational field equations are obtained as a generalization or completion of Newton's dynamical gravitational field equations [10].

5. Motion of Particles of Non-zero Rest Masses

In this section, we study the motion of particles of non-zero rest masses in homogeneous oblate spheroidal space time.

The general relativistic equation of motion for test particles and the coefficients of affine connection for the gravitational field exterior to an oblate spheroidal mass are used to study the motion of particles of non-zero rest masses in this field [2]. The time equation of motion is obtained as

$$\ddot{t} + \frac{2}{c^2}\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\left(\dot{\eta}\frac{\partial f(\eta,\xi)}{\partial\eta} + \dot{\xi}\frac{\partial f((\eta,\xi))}{\partial\xi}\right)\dot{t} = 0 \quad (32)$$

The solution of Eqn. 32 is given as

$$\dot{t} = \left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1} \quad (33)$$

Eqn. 33 is the expression for the variation of the time on a clock moving in this gravitational field. It is of same form as that of Schwarzschild's gravitational field.

Similarly, the η equation of motion is obtained as

$$\ddot{\eta} + \Gamma_{00}^1 c^2\dot{t}^2 + \Gamma_{11}^1\dot{\eta}^2 + \Gamma_{22}^1\dot{\xi}^2 + \Gamma_{33}^1\dot{\phi}^2 + 2\Gamma_{12}^1\dot{\eta}\dot{\xi} = 0 \quad (34)$$

The ξ equation of motion is given as;

$$\ddot{\xi} + \Gamma_{00}^2 c^2\dot{t}^2 + \Gamma_{11}^2\dot{\eta}^2 + \Gamma_{22}^2\dot{\xi}^2 + \Gamma_{33}^2\dot{\phi}^2 + 2\Gamma_{12}^2\dot{\eta}\dot{\xi} = 0 \quad (35)$$

The azimuthal equation of motion is obtained as

$$\ddot{\phi} + 2\Gamma_{13}^3\dot{\eta}\dot{\phi} + 2\Gamma_{23}^3\dot{\eta}\dot{\phi} = 0 \quad (36)$$

The solution of Eqn. 36 is given as

$$\dot{\phi} = \frac{l}{\left(1-\eta^2\right)\left(1+\xi^2\right)} \quad (37)$$

Where, l is a constant of motion. The constant, l physically corresponds to the angular momentum and hence Eqn. 37 is the law of conservation of angular momentum in this gravitational field. It does not depend on the gravitational potential and is of same form as that obtained for Schwarzschild's and Newton's dynamical theory of gravitation.

6. Planetary Motion and Motion of Photons

Here, the gravitational field Lagrangian is constructed and used to study the planetary motion and the motion of photons in the equatorial plane of homogeneous oblate spheroids.

The Lagrangian in the equatorial plane of a homogeneous oblate spheroidal mass is obtained [2] as

$$L=\frac{1}{c}\left[-\left(1+\frac{2}{c^2}f(\xi)\right)\dot{t}^2-\frac{a^2\xi^2}{1+\xi^2}\left(1+\frac{2}{c^2}f(\xi)\right)^{-1}\dot{\xi}^2+a^2(1+\xi^2)\dot{\phi}^2\right]^{\frac{1}{2}} \tag{38}$$

Using the Euler-Lagrange equations for a conservative system in which the potential energy is independent of the generalized velocities, it is shown that

$$\left(1+\frac{2}{c^2}f(\xi)\right)\dot{t}=k, \quad \dot{k}=0 \tag{39}$$

Where, k is a constant. This is the law of conservation of energy in the equatorial plane of the gravitational field exterior to an oblate spheroidal mass [2].

Also,

$$\left(1+\xi^2\right)\dot{\phi}=l, \quad \dot{l}=0 \tag{40}$$

Where, l is a constant. This is the law of conservation of angular momentum in the equatorial plane of the gravitational field exterior to an oblate spheroidal body.

It is well known [12] that the Lagrangian, $L=\in$, with $\in=1$ for time like orbits and $\in=0$ for null orbits. Setting $L=\in$ in Eqn. 38, squaring both sides and substituting Eqns. 39 and 40 yields

$$\frac{a^2\xi^2}{\left(1+\xi^2\right)}\dot{\xi}^2+\frac{a^2l^2}{\left(1+\xi^2\right)}\left(1+\frac{2}{c^2}f(\xi)\right)-2\in^2 f(\xi)=c^2\in^2+1 \tag{41}$$

In most applications of general relativity, the shape of orbits (that is as a function of the azimuthal angle) is of more interest than their time history. Hence, it is instructive to transform Eqn.41 into an equation in terms of the azimuthal angle ϕ. Now, using the transformation; $\xi=\xi(\phi)$ and $u(\phi)=\frac{1}{\xi(\phi)}$ an equivalent form of Eqn. 41 is obtained as

$$\frac{1}{\left(1+u^2\right)^3}\left(\frac{du}{d\phi}\right)^2+\frac{u^2}{1+u^2}\left(1+\frac{2}{c^2}f(u)\right)-\frac{2\in^2 f(u)}{a^2l^2}=\frac{c^2\in^2+1}{a^2l^2} \tag{42}$$

Differentiating Eqn. 42 yields

$$\frac{d^2u}{d\phi^2}-3u\left(1+u^2\right)\frac{du}{d\phi}+\frac{\left(u+u^2\right)}{2}\left(u^2-u+2\right)\left(1+\frac{2}{c^2}f(u)\right)=\left(\frac{1+u^2}{acl}\right)^2\left(a^2c^2l^2-\in^2-\in^2u^2\right)\frac{d}{du}f(u) \tag{43}$$

For time like orbits ($\in=1$), Eqn. 43 reduces to

$$\frac{d^2u}{d\phi^2}-3u\left(1+u^2\right)\frac{du}{d\phi}+\frac{\left(u+u^2\right)}{2}\left(u^2-u+2\right)\left(1+\frac{2}{c^2}f(u)\right)=\left(\frac{1+u^2}{acl}\right)^2\left(a^2c^2u^2-1-u^2\right)\frac{d}{du}f(u) \tag{44}$$

This is the newly derived planetary equation of motion in this gravitational field. It can be solved to obtain the perihelion precision of planetary orbits.

Light rays travels on null geodesics ($\in=0$). Thus Eqn. 43 becomes

$$\frac{d^2u}{d\phi^2}-3u(1+u^2)\frac{du}{d\phi}+\frac{(u+u^2)}{2}(u^2-u+2)\left(1+\frac{2}{c^2}f(u)\right)=\frac{u^2}{c^2}(1+u^2)^2\frac{d}{du}f(u) \tag{45}$$

In the limit of special relativity, some terms in Eqn. 45 vanish and the equation becomes

$$\frac{d^2u}{d\phi^2}-3u\left(1+u^2\right)\frac{du}{d\phi}+\frac{\left(u+u^2\right)}{2}\left(u^2-u+2\right)=0 \tag{46}$$

Eqn. 45 is the photon equation of motion in the vicinity of a static massive homogeneous oblate spheroidal body. For the solution of special relativistic case, Eqn. 46 can be used to solve the general relativistic equation (Eqn. 45). This can be done by taking the general solution of Eqn. 46 to be a perturbation of the solution of Eqn. 45. The immediate consequence of this analysis is that it will produce a new expression for the total deflection of light grazing a massive oblate spheroidal body such as the Sun. This is also open for further research and astrophysical interpretations.

7. Effects of the Oblateness of Sun and Planets on Some Gravitational Phenomena

7.1 Gravitational scalar potential

In this subsection, an expression for the gravitational scalar potential is obtained and values along the equator and pole of the homogeneous oblate spheroidal Sun and planets are computed.

The gravitational scalar potential exterior to a homogeneous oblate spheroid [14] is given as

$$\Phi(\eta,\xi) = B_0 Q_0(-i\xi) + B_2 Q_2(-i\xi) P_2(\eta) \quad (47)$$

Where, Q_0 and Q_2 are the Legendre functions linearly independent to the Legendre polynomials P_0 and P_2, respectively. B_0 and B_2 are constants.

Using Eqn. 47, we obtained [3] approximate expressions for the exterior gravitational scalar potential along the equator and pole of homogeneous oblate spheroids respectively as

$$\Phi(\eta,\xi) \approx \frac{B_0}{3\xi^3}\left(1+3\xi^2\right)i + \frac{B_2}{30\xi^3}\left(7+15\xi^2\right)i \quad (48)$$

and

$$\Phi(\eta,\xi) \approx \frac{B_0}{3\xi^3}\left(1+3\xi^2\right)i - \frac{B_2}{15\xi^3}\left(7+15\xi^2\right)i \quad (49)$$

These two equations were then used to compute approximate values for the gravitational scalar potential along the equator and the pole at various points exterior to the oblate spheroidal bodies in the solar system [3].

7.2 Gravitational time dilation

Here, we consider a clock at rest in this field such that $d\xi = d\eta = d\phi = 0$ and thus the world line element for the gravitational field exterior to an oblate spheroidal mass gives an expression for time dilation in this gravitational field as;

$$d\tau = \left[1 + \frac{2}{c^2} f(\eta,\xi)\right]^{\frac{1}{2}} dt \quad (50)$$

or

$$dt = \left[1 + \frac{2}{c^2} f(\eta,\xi)\right]^{-\frac{1}{2}} d\tau \quad (51)$$

Expanding the right-hand side of Eqn.51 gives

$$dt = \left[1 - \frac{1}{c^2} f(\eta,\xi) + \ldots\right] d\tau \quad (52)$$

From Eqn. 52, it can be conveniently deduced that $dt > d\tau$ (dilation). Thus, coordinate time of a clock in this gravitational field is dilated relative to proper time [4].

As an illustration (Table 1), consider two events at fixed points exterior to planet Earth along the equator, separated in this gravitational field by coordinate time dt and proper time $d\tau$.

Table 1: Coordinate time along the equator in the gravitational field exterior to the Earth as a factor of proper time [4].

Fixed point along the equator	Radial distance along the equator (km)	dt as factor of $d\tau$
ξ_0	6,378	1.306170
$2\xi_0$	12,723	1.122655
$3\xi_0$	19,075	1.076871
$4\xi_0$	25,430	1.055996
$5\xi_0$	31,784	1.044042
$6\xi_0$	38,140	1.036296
$7\xi_0$	44,495	1.030867
$8\xi_0$	50,851	1.026852
$9\xi_0$	57,207	1.023761
$10\xi_0$	63,562	1.021308

Hence, one concludes that clocks run more slowly at a smaller distance from the massive oblate spheroidal body. In other words, clocks will run slower at lower gravitational potentials (deeper within a gravity well). This was first confirmed experimentally in the laboratory by the Hafele-Keating [15] experiment. Today, there are numerous direct measurements of gravitational time dilation using atomic clocks, while ongoing validation is provided as a side-effect of the operation of Global Positioning Systems [16].

7.3 Gravitational spectral shift

Here, a beam of light moving from a source or emitter (E) at a fixed point in the gravitational field of the oblate spheroidal body to an observer or receiver (R) at a fixed point in the same gravitational field is considered. Einstein's equation of motion for a photon is used to derive an expression for the shift in the frequency of a photon moving in the gravitational field of an oblate spheroidal mass as:

$$\frac{\nu_R}{\nu_E} = \frac{\Delta\tau_E}{\Delta\tau_R} =$$

$$\left(1+\frac{2}{c^2} f_E(\eta,\xi)\right)^{\frac{1}{2}}\left(1+\frac{2}{c^2} f_R(\eta,\xi)\right)^{-\frac{1}{2}} \quad (53)$$

Where, ν_R and ν_E are the frequencies of the received and emitted photons, respectively. Also, $\Delta\tau_R$ and $\Delta\tau_E$ are the respective proper time intervals between two light signals at reception and emission points. The expressions on the right-hand side of Eqn. 53 are converging and can be expanded binomially to the order of c^{-2} in approximate gravitational fields. This gives

$$z \equiv \frac{\Delta\nu}{\nu_E} \equiv \frac{\nu_R - \nu_E}{\nu_E} \approx \frac{1}{c^2}\left(f_E(\eta,\xi) - f_R(\eta,\xi)\right) \quad (54)$$

It follows from Eqn. 54 that if the source is nearer to a body than the receiver, then $f_E(\eta,\xi) < f_R(\eta,\xi)$ and hence $\Delta\nu < 0$.

This indicates that there is a reduction in the frequency of light when the source or emitter is nearer the body than the receiver. The light is said to have undergone a red shift. Otherwise (source further away from body than receiver), the light undergoes a blue shift [5].

This gravitattional phenomena was experimentally confirmed in the laboratory by the Pound-Rebka experiment in 1959 (they used the Mossbauer effect to measure the change in frequency in gamma rays as they travelled from the ground to the top of Jefferson Labs at Havard University). The effect of a gravitational potential difference on the apparent energy of the 14.4 keV gamma ray of Fe^{57} was found by Pound and Rebka [17] to agree within uncertainties, with Einstein's prediction based on his principle of equivalence. Pound and Rebka in 1964 improved on their earlier results confirming Einstein's prediction to greater precision. The resonance of the 14.4 keV Fe^{57} gamma ray between Iron foils was still employed. The same height as in the earlier experiment in the Jefferson Physical Laboratory (22.5m) was also used. This gravitational phenomenon was later confirmed by astronomical observations.

Now, suppose the Pound-Rebka experiment was performed at the surface of the Earth on the equator, the shift in frequency obtained is: $z \cong 2.578 \text{ x } 10^{-15}$. This value is quite close to that obtained by Pound and Rebka ($z \cong 2.45 \text{ x } 10^{-15}$) in 1964. The closeness of the theoretically computed value for the Pound-Rebka experiment is remarkable indeed. The gravitational spectral shift for the Pound-Rebka experiment is predicted for other oblate spheroidal astrophysical bodies in the solar system if it is to be performed along the equator (Table 2).

Table 2: Predicted Pound-Rebka shift in frequency for some astrophysical bodies in the Solar System [5].

Body	Observation distance (km)	ξ at point	f_R (Nmkg^{-1})	Predicted shift
Sun	700,022.5	241.527	-1.9373218 x 10^{11}	-2.85889 x 10^{-21}
Mars	3418.5	9.231	-1.2317966 x 10^{7}	-9.24256 x 10^{-20}
Jupiter	71512.5	2.641	-1.4958977 x 10^{9}	-1.010111 x 10^{-20}
Saturn	60292.5	1.971	-4.8484869 x 10^{8}	-1.902222 x 10^{-21}
Uranus	25582.5	3.994	-2.1522082 x 10^{8}	-4.647889 x 10^{-20}
Neptune	24782.5	4.304	-2.5196722 x 10^{8}	-5.168667 x 10^{-20}

8. Conclusion

The practicability of the findings in this work is an encouraging factor. More so, the application of these results is another factor in this age of computational precision. The astrophysical applications of our extension abound as all applications of Schwarzschild's metric in studying gravitational phenomena in the solar system can now be studied using the metric tensor in an oblate spheroidal gravitational field.

Appendix: Field equations in terms of the metric tensor and affine connections

$$\begin{aligned}0 = &-\Gamma^1_{00,1} + \Gamma^0_{01}\Gamma^1_{00} - \Gamma^1_{00}\Gamma^1_{11} - \Gamma^2_{00}\Gamma^1_{21} - \Gamma^2_{00,2} + \Gamma^0_{02}\Gamma^2_{00} \\ &- \Gamma^1_{00}\Gamma^2_{12} - \Gamma^2_{00}\Gamma^2_{22} - \Gamma^1_{00}\Gamma^3_{13} - \Gamma^2_{00}\Gamma^3_{23} - \frac{1}{2}Rg_{00}\end{aligned} \tag{A1}$$

$$\begin{aligned}0 = &\Gamma^0_{10,1} - \Gamma^1_{11}\Gamma^0_{10} - \Gamma^2_{11}\Gamma^0_{20} + \left(\Gamma^0_{01}\right)^2 + \Gamma^2_{12,1} - \Gamma^2_{11,2} + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^1_{11}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} \\ &+ \Gamma^3_{13,1} + \Gamma^3_{13}\Gamma^3_{31} - \Gamma^1_{11}\Gamma^3_{13} - \Gamma^2_{11}\Gamma^3_{23} - \frac{1}{2}Rg_{11}\end{aligned} \tag{A2}$$

$$\begin{aligned}0 = &\Gamma^0_{10,2} - \Gamma^1_{12}\Gamma^0_{10} - \Gamma^2_{12}\Gamma^0_{20} + \Gamma^0_{10}\Gamma^0_{02} - \Gamma^1_{12,1} + \Gamma^1_{11,2} - \Gamma^2_{12}\Gamma^1_{21} + \Gamma^2_{11}\Gamma^1_{22} \\ &+ \Gamma^3_{13,2} + \Gamma^3_{13}\Gamma^3_{32} - \Gamma^1_{12}\Gamma^3_{13} - \Gamma^2_{12}\Gamma^3_{23} - \frac{1}{2}Rg_{12}\end{aligned} \tag{A3}$$

$$\begin{aligned}0 = &\Gamma^0_{20,2} - \Gamma^1_{22}\Gamma^0_{10} - \Gamma^2_{22}\Gamma^0_{20} + \Gamma^0_{20}\Gamma^0_{02} - \Gamma^1_{22,1} + \Gamma^1_{12,2} - \Gamma^1_{22}\Gamma^1_{11} - \Gamma^2_{22}\Gamma^1_{21} + \Gamma^1_{21}\Gamma^1_{12} \\ &+ \Gamma^3_{23,2} + \Gamma^3_{23}\Gamma^3_{32} - \Gamma^1_{22}\Gamma^3_{13} - \Gamma^2_{22}\Gamma^3_{23} - \frac{1}{2}Rg_{22}\end{aligned} \tag{A4}$$

$$\begin{aligned}0 = &-\Gamma^1_{33}\Gamma^0_{10} - \Gamma^2_{33}\Gamma^0_{20} - \Gamma^1_{33,1} - \Gamma^1_{33}\Gamma^1_{11} - \Gamma^2_{33}\Gamma^1_{21} + \Gamma^3_{31}\Gamma^1_{33} \\ &- \Gamma^2_{33,2} - \Gamma^1_{33}\Gamma^2_{12} - \Gamma^2_{33}\Gamma^2_{22} + \Gamma^3_{32}\Gamma^2_{33} - \frac{1}{2}Rg_{33}\end{aligned} \tag{A5}$$

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