Motion of Test Particles and Orbits Exterior to Static Homogeneous Prolate Spheroidal Space-Time

Chifu Ebenezer Ndikilar*
Department of Physics, Gombe State University, P.M.B 127, Gombe, Nigeria

We use a newly constructed metric tensor exterior to homogeneous prolate spheroidal mass distributions to derive equations of motion for particles of non-zero rest masses. The time equation of motion gives us an expression for the variation of the time on a clock moving in this gravitational field. From the azimuthal equation of motion, we obtained an expression for the law of conservation of angular momentum in this field. We then construct the Lagrangian for this gravitational field and use it to study orbits in the equatorial plane of a homogenous prolate spheroidal mass.

1. Introduction

The term 'prolate spheroid' refers to the shape of some moons in the solar system. Examples of these moons are Mimas, Enceladus and Tethys of Saturn, and Miranda of Uranus. The prolate spheroidal geometry is also used to describe the shape of some nebulae. A nebula is a region or cloud of interstellar dust and gas appearing variously as a hazy bright or dark patch. The Crab Nebula is one such well-known example [1,2]. Also, the existence of rotating prolate spheroidal galaxies has been known for decades, yet, a theoretical model based on Newton's or Einstein's gravitational theories remains elusive [3].

We concentrate in this article on gravitational sources with time-independent and axially-symmetric distributions of mass within spheroids, which are characterized by at most two typical integrals of geodesic motion: energy and angular momentum. From an astrophysical point of view, such an assumption could prove useful because it is equivalent to the assumption that the gravitational source is changing slowly in time so that partial time derivatives are negligible compared to the spatial ones. We stress that the mass source considered is not the most arbitrary one from a theoretical point of view, but on the other hand, many astrophysically interesting systems are usually assumed to be time independent (or static from another point of view) and axially symmetric continuous sources [4].

*ebenechifu@yahoo.com

2. Metric tensor and Coefficients of Affine Connection

It has been shown [5] that the covariant metric tensor exterior to static homogeneous prolate spheroidal distributions of mass is given as

$$g_{00} = \left(1 + \frac{2}{c^2} f(\eta, \xi)\right) \tag{2.1}$$

$$g_{11} = -\frac{a^2\eta^2}{\eta^2 + \xi^2 - 1}\left[\left(1 + \frac{2}{c^2} f(\eta, \xi)\right)^{-1} + \frac{\xi^2(1-\xi^2)}{\eta^2(\eta^2 - 1)}\right] \tag{2.2}$$

$$g_{12} = g_{21} = -\frac{a^2\eta\xi}{\eta^2 + \xi^2 - 1}\left[-1 + \left(1 + \frac{2}{c^2} f(\eta, \xi)\right)^{-1}\right] \tag{2.3}$$

$$g_{22} = -\frac{a^2\xi^2}{\eta^2 + \xi^2 - 1}\left[\left(1 + \frac{2}{c^2} f(\eta, \xi)\right)^{-1} + \frac{\eta^2(\eta^2 - 1)}{\xi^2(1-\xi^2)}\right] \tag{2.4}$$

$$g_{33} = -a^2\left(1-\xi^2\right)\left(\eta^2 - 1\right) \tag{2.5}$$

$$g_{\mu\nu} = 0; \quad \text{otherwise} \tag{2.6}$$

This metric tensor has the same number of non-zero components (six) as the metric exterior to an oblate spheroid [6]. It has been noted that in the case of oblate spheroids, $f(\eta, \xi)$ is an arbitrary function determined by the mass or pressure and hence possesses *a priori* all the symmetries of the latter. The function $f(\eta,\xi)$ can be approximated to be equal to Newton's gravitational scalar potential exterior to the mass distribution. The metric tensors by virtue of their construction satisfy the first and second postulates of General relativity. There are invariance of the line element and Einstein's gravitational field equations [5-7].

After the evaluation of the cofactors of covariant tensor, the contravariant metric tensor exterior to a prolate spheroidal mass is found to be given explicitly in prolate spheroidal coordinates as

$$g^{00} = \left[1 + \frac{2}{c^2} f(\eta,\xi)\right]^{-1} \tag{2.7}$$

$$g^{11} = \frac{-(\eta^2-1)(\eta^2+\xi^2-1)\left[\eta^2(\eta^2-1)+\xi^2(1+\xi^2)\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\right]}{a^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\left[\eta^2(\eta^2-1)-\xi^2(1-\xi^2)\right]^2} \tag{2.8}$$

$$g^{12} \equiv g^{21} = \frac{-\eta\xi(\eta^2-1)(1-\xi^2)(\eta^2+\xi^2-1)\left[-1+\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\right]}{a^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\left[\eta^2(\eta^2-1)-\xi^2(1-\xi^2)\right]^2} \tag{2.9}$$

$$g^{22} = \frac{-(1-\xi^2)(\eta^2+\xi^2-1)\left[\xi^2(1-\xi^2)+\eta^2(1-\eta^2)\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\right]}{a^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\left[\eta^2(1-\eta^2)-\xi^2(1-\xi^2)\right]^2} \tag{2.10}$$

$$g^{33} = -\left[a^2(1-\xi^2)(\eta^2-1)\right]^{-1} \tag{2.11}$$

$$g^{\mu\nu} = 0; \quad \text{otherwise} \tag{2.12}$$

The contravariant metric tensor has two additional non-zero components not found in Schwarzschild field. Notice that unlike the Schwarzschild field, where all the non-zero components of the contravariant tensor are simply reciprocals of the covariant metric tensor, here only Eqns. 2.7 and 2.12 are reciprocals of their respective covariant tensors. The other non-zero components have a common denominator.

The coefficients of affine connection for the gravitational field exterior to a static homogeneous prolate spheroidal mass is found to be given in terms of the metric tensors for the gravitational field as

$$\Gamma^0_{01} \equiv \Gamma^0_{10} = \frac{1}{2} g^{00} g_{00,1} \tag{2.13}$$

$$\Gamma^0_{02} \equiv \Gamma^0_{20} = \frac{1}{2} g^{00} g_{00,2} \tag{2.14}$$

$$\Gamma^1_{00} = -\frac{1}{2}\left(g^{11} g_{00,1} + g^{12} g_{00,2}\right) \tag{2.15}$$

$$\Gamma^1_{11} = \frac{1}{2}\left(g^{11} g_{11,1} + g^{12}\left(2g_{12,1} - g_{11,2}\right)\right) \tag{2.16}$$

$$\Gamma^1_{12} \equiv \Gamma^1_{21} = \frac{1}{2}\left(g^{11} g_{11,2} + g^{12} g_{22,1}\right) \tag{2.17}$$

$$\Gamma^1_{22} = \frac{1}{2}\left(g^{11}\left(2g_{12,2} - g_{22,1}\right) + g^{12} g_{22,2}\right) \tag{2.18}$$

$$\Gamma^1_{33} = -\frac{1}{2}\left(g^{11} g_{33,1} + g^{12} g_{33,2}\right) \tag{2.19}$$

$$\Gamma^2_{00} = -\frac{1}{2}\left(g^{21} g_{00,1} + g^{22} g_{00,2}\right) \tag{2.20}$$

$$\Gamma^2_{11} = \frac{1}{2}\left(g^{21} g_{11,1} + g^{22}\left(2g_{12,1} - g_{11,2}\right)\right) \tag{2.21}$$

$$\Gamma^{2}_{12} \equiv \Gamma^{2}_{21} = \frac{1}{2}\left(g^{21}g_{11,2} + g^{22}g_{22,1}\right) \quad (2.22)$$

$$\Gamma^{2}_{22} = \frac{1}{2}\left(g^{21}\left(2g_{12,2} - g_{22,1}\right) + g^{22}g_{22,2}\right) \quad (2.23)$$

$$\Gamma^{2}_{33} = -\frac{1}{2}\left(g^{21}g_{33,1} + g^{22}g_{33,2}\right) \quad (2.24)$$

$$\Gamma^{3}_{13} = \Gamma^{3}_{31} = \frac{1}{2}g^{33}g_{33,1} \quad (2.25)$$

$$\Gamma^{3}_{23} = \Gamma^{3}_{32} = \frac{1}{2}g^{33}g_{33,2} \quad (2.26)$$

$$\Gamma^{\delta}_{\alpha\beta} = 0; \qquad \text{otherwise} \quad (2.27)$$

3. Equations of Motion of Particles of Non-Zero Rest Masses

The general relativistic equation of motion for particles of non-zero rest masses in a gravitational field are given by [8]

$$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\nu\lambda}\left(\frac{dx^{\nu}}{d\tau}\right)\left(\frac{dx^{\lambda}}{d\tau}\right) = 0 \quad (3.1)$$

The time, eta, zeta and phi general relativistic equations of motion for test particles can be evaluated as follows.

Setting, $\mu = 0$, Eqn. 3.1 gives

$$\frac{d^2x^0}{d\tau^2} + 2\Gamma^{0}_{01}\frac{dx^0}{d\tau}\frac{dx^1}{d\tau} + 2\Gamma^{0}_{02}\frac{dx^0}{d\tau}\frac{dx^2}{d\tau} = 0 \quad (3.2)$$

Denoting differentiation with respect to time as dot (.) in Eqn. 3.2 yields

$$\ddot{t} + 2\Gamma^{0}_{01}\dot{t}\dot{\eta} + 2\Gamma^{0}_{02}\dot{t}\dot{\xi} = 0 \quad (3.3)$$

Substituting the explicit expressions for the affine connections in Eqn. 3.3 gives

$$\ddot{t} + \frac{2}{c^2}\left(1 + \frac{2}{c^2}f(\eta,\xi)\right)^{-1}$$

$$\left(\dot{\eta}\frac{\partial f(\eta,\xi)}{\partial\eta} + \dot{\xi}\frac{\partial f((\eta,\xi))}{\partial\xi}\right)\dot{t} = 0 \quad (3.4)$$

Or,

$$\frac{\ddot{t}}{\dot{t}} + \frac{d}{d\tau}\left[\ln\left(1 + \frac{2}{c^2}f(\eta,\xi)\right)\right] = 0 \quad (3.5)$$

Or,

$$\frac{d}{d\tau}\left(\ln\dot{t}\right) + \frac{d}{d\tau}\left[\ln\left(1 + \frac{2}{c^2}f(\eta,\xi)\right)\right] = 0 \quad (3.6)$$

Integrating Eqn. 3.6 yields

$$\ln\dot{t}\left(1 + \frac{2}{c^2}f(\eta,\xi)\right) = A \quad constant \quad (3.7)$$

Or,

$$\dot{t} = A\left(1 + \frac{2}{c^2}f(\eta,\xi)\right)^{-1} \quad (3.8)$$

As, $t \to \tau$, $f(\eta,\xi) \to 0$ and the constant $A \equiv 1$. Thus,

$$\dot{t} = \left(1 + \frac{2}{c^2}f(\eta,\xi)\right)^{-1} \quad (3.9)$$

Eqn. 3.9 is the expression for time variation on a clock moving in this gravitational field. It is of the same form as that obtained in the oblate spheroidal gravitational field and in Schwarzschild's field [6].

Similarly, setting $\mu = 1$ in Eqn. 3.1 and expanding, we obtain the η equation of motion as

$$\ddot{\eta} + \Gamma^{1}_{00}c^2\dot{t}^2 + \Gamma^{1}_{11}\dot{\eta}^2 + \Gamma^{1}_{22}\dot{\xi}^2 + \Gamma^{1}_{33}\dot{\phi}^2 + 2\Gamma^{1}_{12}\dot{\eta}\dot{\xi} = 0 \quad (3.10)$$

For pure η motion and $\dot{\xi} = \dot{\phi} = 0$, Eqn. 3.10 reduces to

$$\ddot{\eta} + \Gamma^{1}_{00}c^2\dot{t}^2 + \Gamma^{1}_{11}\dot{\eta}^2 = 0 \quad (3.11)$$

Or,

$$\ddot{\eta} + \Gamma^{1}_{11}\dot{\eta}^2 + c^2\Gamma^{1}_{00}\left(1 + \frac{2}{c^2}f(\eta,\xi)\right)^{-1} = 0 \quad (3.12)$$

The pure η equation of motion in this field is also a second order nonlinear differential equation.

Similarly, it can be shown that the ξ equation of motion is given as

$$\ddot{\xi} + \Gamma^2_{00} c^2 \dot{t}^2 + \Gamma^2_{11} \dot{\eta}^2 + \Gamma^2_{22} \dot{\xi}^2 + \Gamma^2_{33} \dot{\phi}^2 + 2\Gamma^2_{12} \dot{\eta}\dot{\xi} = 0 \quad (3.13)$$

For pure ξ motion and $\dot{\eta} = \dot{\phi} = 0$, Eqn. 3.13 reduces to

$$\ddot{\xi} + \Gamma^2_{22} \dot{\xi}^2 + c^2 \Gamma^2_{00} \left(1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-2} = 0 \quad (3.14)$$

For azimuthal motion, we set $\mu = 3$ in Eqn. 3.1 to obtain the azimuthal equation of motion as

$$\ddot{\phi} + 2\Gamma^3_{13} \dot{\eta}\dot{\phi} + 2\Gamma^3_{23} \dot{\eta}\dot{\phi} = 0 \quad (3.15)$$

Or,

$$\frac{\ddot{\phi}}{\dot{\phi}} + \frac{2\eta}{\eta^2 - 1} \dot{\eta} - \frac{2\xi}{1 - \xi^2} \dot{\xi} = 0 \quad (3.16)$$

Or,

$$\frac{d}{d\tau} \left(\ln \dot{\phi} \right) + \frac{d}{d\tau} \left(\ln \left(\eta^2 - 1 \right) \left(1 - \xi^2 \right) \right) = 0 \quad (3.17)$$

Thus, the azimuthal equation of motion for our gravitational field is given as

$$\dot{\phi} = \frac{l}{\left(\eta^2 - 1 \right) \left(1 - \xi^2 \right)} \quad (3.18)$$

Where, l is a constant of motion, which physically corresponds to the angular momentum and hence Eqn. 3.18 is the law of conservation of angular momentum in this gravitational field [9]. It has the same form as that obtained in the oblate spheroidal gravitational field [6] and does not depend on the gravitational potential. Therefore, it is of same form as that obtained in Schwarzschild's and Newton's dynamical theory of gravitation. The significance of these results is that the law of conservation of angular momentum takes the same form in three different gravitational fields and thus the expression for this law of mechanics is invariant with respect to three gravitational fields

4. Orbits in Homogeneous Prolate Spheroidal Space Time

The Lagrangian in the space time exterior to a prolate spheroid is defined [9] as

$$L = \frac{1}{c} \left(-g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \right)^{\frac{1}{2}} \quad (4.1)$$

The above Lagrangian can be written more explicitly in prolate spheroidal coordinates as

$$L = \frac{1}{c} \left(-g_{00} \left(\frac{dt}{d\tau} \right)^2 - g_{11} \left(\frac{d\eta}{d\tau} \right)^2 - 2g_{12} \left(\frac{d\eta}{d\tau} \right) \left(\frac{d\xi}{d\tau} \right) - g_{22} \left(\frac{d\xi}{d\tau} \right)^2 - g_{33} \left(\frac{d\phi}{d\tau} \right)^2 \right)^{\frac{1}{2}} \quad (4.2)$$

For orbits in the equatorial plane of a homogeneous prolate spheroidal mass, $\eta \equiv 1$ and thus the Lagrangian (Eqn. 4.2) reduces to

$$L = \frac{1}{c} \left(-g_{00} \left(\frac{dt}{d\tau} \right)^2 - g_{22} \left(\frac{d\xi}{d\tau} \right)^2 - g_{33} \left(\frac{d\phi}{d\tau} \right)^2 \right)^{\frac{1}{2}} \quad (4.3)$$

Substituting the explicit expressions for the components of the covariant metric tensor in the equatorial plane yields

$$L = \frac{1}{c} \left[-\left(1 + \frac{2}{c^2} f(\xi) \right) \left(\frac{dt}{d\tau} \right)^2 + a^2 \left(1 + \frac{2}{c^2} f(\xi) \right)^{-1} \left(\frac{d\xi}{d\tau} \right)^2 \right]^{\frac{1}{2}} \quad (4.4)$$

Or,

$$L = \frac{1}{c}\left[-\left(1+\frac{2}{c^2} f(\xi)\right)\dot{t}^2 + a^2\left(1+\frac{2}{c^2} f(\xi)\right)^{-1} \dot{\xi}^2 \right]^{\frac{1}{2}} \tag{4.5}$$

It is well known that the gravitational field is a conservative field. Therefore, the Euler-Lagrange equations for a conservative system in which the potential energy is independent of the generalized velocities [9] is written as,

$$\frac{\partial L}{\partial x^{\alpha}} = \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^{\alpha}}\right) \tag{4.6}$$

But

$$\frac{\partial L}{\partial x^0} \equiv \frac{\partial L}{\partial t} = 0 \tag{4.7}$$

Thus, from Eqn. 4.6, we deduce that

$$\frac{\partial L}{\partial \dot{t}} = constan\,t \tag{4.8}$$

From Eqn. 4.5, it can be shown using Eqn. 4.8 that

$$\left(1+\frac{2}{c^2} f(\xi)\right)\dot{t} = k, \quad \dot{k} = 0 \tag{4.9}$$

Where, k is a constant. This is the law of conservation of energy in the equatorial plane of the gravitational field exterior to a prolate spheroidal mass. Notice that Eqn. 4.9 is exactly the same as the expression obtained from the time equation of motion for test particles (Eqn. 3.9). Instructively, it is of the same form as the expression in Schwarzschild's and oblate spheroidal gravitational fields [6]. Thus we arrive at the conclusion that the law of conservation of energy takes the same form in all the three gravitational fields. Hence, this physical law is invariant of gravitational field.

It is well known [9] that the Lagrangian $L = \in$, with $\in = 1$ for time like orbits and $\in = 0$ for null orbits. Setting $L = \in$ in Eqn. 4.5 and squaring both sides yields

$$\in^2 = -\frac{1}{c^2}\left(1+\frac{2}{c^2} f(\xi)\right)\dot{t}^2 + \frac{a^2}{c^2}\left(1+\frac{2}{c^2} f(\xi)\right)^{-1} \dot{\xi}^2 \tag{4.10}$$

Substituting Eqn. 4.9 into Eqn. 4.10 and simplifying, we get

$$\in^2 = -\frac{1}{c^2}\left(1+\frac{2}{c^2} f(\xi)\right)^{-1} + \frac{a^2}{c^2}\left(1+\frac{2}{c^2} f(\xi)\right)^{-1} \dot{\xi}^2 \tag{4.11}$$

Or,

$$a^2\dot{\xi}^2 - 2\in^2 f(\xi) = c^2\in^2 + 1 \tag{4.12}$$

In most applications of general relativity, it is well known that we are more interested in the shape of orbits (as a function of the azimuthal angle) than in their time history [8]. Hence, it is instructive to transform Eqn. 4.12 into an equation in terms of the azimuthal angle ϕ. Now, let us consider the following transformation with $\xi = \xi(\phi)$ and $u(\phi) = \dfrac{1}{\xi(\phi)}$

Then

$$\dot{\xi} = \dot{\phi}\frac{d\xi}{d\phi} \quad \text{or} \quad \dot{\xi} = \frac{l}{1+\xi^2}\frac{d\xi}{d\phi} \tag{4.13}$$

But,

$$\frac{d\xi}{d\phi} = \frac{d\xi}{du}\frac{du}{d\phi} \quad \text{or} \quad \frac{d\xi}{d\phi} = -u^{-2}\frac{du}{d\phi} \tag{4.14}$$

Thus,

$$\dot{\xi} = -\frac{l}{1+u^2}\frac{du}{d\phi} \tag{4.15}$$

Now, imposing the transformation Eqns. 4.13 and 4.15 on Eqn. 4.12 and simplifying, we get

$$\frac{1}{\left(1+u^2\right)^2}\left(\frac{du}{d\phi}\right)^2 - \frac{2\in^2 f(u)}{a^2 l^2} = \frac{c^2\in^2 + 1}{a^2 l^2} \tag{4.16}$$

To obtain a second order differential equation, we differentiate Eqn. 4.16 once, giving us

$$\left(1+u^2\right)^{-2}\frac{d^2u}{d\phi^2}-2u\left(1+u^2\right)^{-3}\frac{du}{d\phi}+\left(\frac{\in}{acl}\right)^2\frac{df}{du}=0 \qquad (4.17)$$

For time like orbits $(\in = 1)$ Eqn. 4.17 reduces to

$$\frac{d^2u}{d\phi^2}-\frac{2u}{1+u^2}\frac{du}{d\phi}+\left(\frac{1+u^2}{acl}\right)^2\frac{df}{du}=0 \qquad (4.18)$$

This is the planetary equation of motion. It can be solved to obtain the perihelion precision of planetary orbits.

Since light rays travel on null geodesics, we have $(\in = 0)$. So Eqn. 4.17 becomes

$$\frac{d^2u}{d\phi^2}-\frac{2u}{1+u^2}\frac{du}{d\phi}=0 \qquad (4.19)$$

Eqn. 4.19 is the photon equation of motion in the vicinity of a static massive homogenous prolate spheroidal body.

5. Conclusion

Our approach is convenient as our metric tensor has only one unknown function, which can be conveniently approximated to the gravitational scalar potential exterior to a homogeneous oblate spheroid. We obtained new expressions for the conservation laws of energy and angular momentum. Our derived planetary equation of motion and the equation of motion of a photon in this gravitational field can be used to study the behavior of particles and light as they orbit prolate spheroidal bodies.

References

[1] V. L. Trimble, Astronomical Society of the Pacific **85**, 507 (1979).

[2] Wikipedia, *The Free Encyclopedia-General Relativity*, http://www.en.wikipedia.org

[3] P. S. Florides and N. K. Spyrou, Astrophysical Journal **419**, 541 (1993).

[4] N. K. Spyrou, Facta Universitatis **1(4)**, 7 (1997).

[5] S. X. K. Howusu, The 210 Astrophysical Solutions plus 210 Cosmological Solutions of Einstein's Gravitational Field Equations, Natural Philosophy Society, Jos, 47 (2007).

[6] E. N. Chifu, A. Usman and O. C. Meludu, Progress in Physics **3**, 49 (2009).

[7] E. N. Chifu, Progress in Physics **4**, 73 (2009).

[8] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (J. Wiley, New York, 1972) p.175.

[9] K. S. D. Peter, *An Introduction to Tensors and Relativity* (Cape Town University Press, 2000) p.51.

Received: 23 November, 2010
Accepted: 03 January, 2011