Fractional and Odd Non-linear Operators in Complex SUSY Hamiltonians
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We generate supersymmetric quantum systems consisting of odd and fractional operators in complex space. Present construction is practically impossible using pure Hermitian operators. Matrix diagonalization method has been used to calculate real eigenvalues. We notice the well behaved nature of wave function mod square corresponding to real eigenvalues of the generated operators.

1. Introduction
Space-time invariance models under the name of PT-symmetry gained sufficient interest among authors working theoretically as well as experimentally during the last two decades after the seminal work of Bender and Boettecher [1]. Here, $P$ is the parity operator having the property

$$P^{-1}xP = -x\quad (1)$$
$$P^{-1}pP = -p\quad (2)$$
$$P^{-1}iP = i\quad (3)$$

Similarly time reversal operator, $T$ has the following property

$$T^{-1}xT = x\quad (4)$$
$$T^{-1}pT = -p\quad (5)$$
$$T^{-1}iT = -i\quad (6)$$

In the above, $x$ is the co-ordinate operator and $p$ is the corresponding momentum operator. The commutation relation between $x$ and $p$ is

$$[x, p] = i\quad (7)$$

The importance of this PT-symmetry model was probably due to unbroken real spectra in complex cubic oscillator [1]

$$H = p^2 + ix^3\quad (8)$$

Further, Bender et al. [2] has reported that operator of the type

$$H = p^2 + (ix)^a |x| ^b\quad (9)$$

with the typical choice of constants $a = \frac{1}{2}$ and $b = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ would yield real spectra. Similarly many theoretical analysis on PT operators motivated many people to visualise the outcome of PT-symmetry system through the experiment. Few interesting experimental observations are as follows.

In optics complex refractive index, $n(x)$ in PT-symmetry means

$$n(x) = (n(-x))^*\quad (10)$$

Where, real part must be even function of $x$ and complex part, $\text{Im}[n(x)]$ (denoting loss or gain) must be odd function of $x$ [3]. This motivates for the development of optical materials. Experiments involving loss and gain analysis in optical wave guides [4-6] have confirmed theoretically. Similarly other experimental findings on synthetic photon lattices [7, 8], optical micro-cavities [9, 10] etc. have nicely agreed with theoretical predictions. Apart from the above, we should not ignore the lack of experimental evidence on supersymmetry (SUSY) partners of quarks, leptons and gauge bosons which strongly suggest that if SUSY exists, it has been spontaneously broken in nature. Hence conventional SUSY [11-14] involving Hermitian operators may not be sufficient to explore new things. Moreover, we do not find any literature on SUSY involving fractional operators involving the term $(ix)^d$ with

$$d = \frac{1}{2}, \frac{3}{2}, \ldots$$

Hence, the aim of the present paper is to generate SUSY operators involving fractional operators as described in the following.

2. Complex SUSY Hamiltonians involving odd and fractional odd operators
Here we use standard formulation on SUSY as [11-14]

$$H^+ = p^2 + \frac{dV}{dx} + V^2\quad (11a)$$
$$H^- = p^2 - \frac{dV}{dx} + V^2\quad (11b)$$

The corresponding eigenvalues must satisfy the relations as

$$E_n^{(+)} = E_{n+1}^{(-)}\quad (12)$$
After so much study on SUSY, we feel not to repeat the above relations [11-14]. In this work, we propose the super-potential as

\[ V(x) = \lambda (ix^3)^{\frac{1}{2}} \]

and the corresponding Hamiltonians are

\[ H^- = p^2 - 1.5\lambda (ix)^{\frac{1}{2}} + \lambda^2 ix^3 \]

\[ H^+ = p^2 + 1.5\lambda (ix)^{\frac{1}{2}} + \lambda^2 ix^3 \]

3. Eigenvalue Calculation

We solve the eigenvalue relations

\[ H|\psi_m\rangle = E_m |\psi_m\rangle \]

using matrix the diagonalization method [15-17]. In the above equation (Eqn. (16)),

\[ |\psi_m\rangle = \sum_k A^{(m)}_k |k\rangle \]

and \( |k\rangle \) is the Harmonic oscillator base function, which satisfy the following condition:

\[ H_0 |k\rangle = \left( p^2 + x^2 \right) |k\rangle = (2k + 1) |k\rangle \]

with \( k = 0, 1, 2, \ldots \). In its explicit form, \( |k\rangle \) can be written as

\[ |k\rangle = \frac{(a^+)^k}{\sqrt{k!}} |0\rangle \]

with \( a|0\rangle = 0 \)

More explicitly,

\[ |0\rangle = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{2}} \]

and \( [a, a^+] = 1 \)

Here one has to solve a recurrence relation involving \( A^{(m)}_k \) as discussed in the literature [12, 13, 16, 17] to find the eigenvalues of the Hamiltonian. The computed first four eigenvalues of \( H^- \) (Eqn. 15(a)) and \( H^+ \) (Eqn. 15(b)) for \( \lambda = 0.01 \) are given in Table 1. Further, in order to ascertain the nature of real eigenvalues, we plot mod square of the wave functions, \( |\psi_{m=0,1}\rangle^2 \) for \( H^- \) and \( H^+ \) operators in Fig. 1 and Fig. 2, respectively.

Table 1. Energy levels of \( PT \) symmetric SUSY complex operators (\( H^- \) (Eqn. 15(a)) and \( H^+ \) (Eqn. 15(b))).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( E_n^{(-)} )</th>
<th>( E_n^{(+))}</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.390 6</td>
</tr>
<tr>
<td>1</td>
<td>0.390 5</td>
<td>0.911 5</td>
</tr>
<tr>
<td>2</td>
<td>0.911 5</td>
<td>1.490 7</td>
</tr>
<tr>
<td>3</td>
<td>1.490 4</td>
<td>2.080 1</td>
</tr>
</tbody>
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Fig. 1. \( |\psi_{m=0,1}\rangle^2 \) of \( H^- \) with \( \lambda = 0.01 \) for (a) \( m=0 \) and (b) \( m=1 \).

Fig. 2. \( |\psi_{m=0,1}\rangle^2 \) of \( H^+ \) with \( \lambda = 0.01 \) for (a) \( m=0 \) and (b) \( m=1 \).

4. Discussions and Conclusion

We notice that one can construct new model SUSY operators using odd and fractional operators. Present construction is new and can hardly be seen in earlier literature [11-14]. The well behaved nature of wave function mod square indicates the existence of real eigenvalues. Interestingly, present SUSY analysis involving fractional and odd operators can hardly be
realised in Hermitian operators. Lastly, we believe present analysis will motivate people to investigate further on finding new SUSY partners, which could be of interest from both theoretical and experimental point of view.

References


Received: 25 June, 2018

Received (revised version): 29 October, 2019

Accepted: 01 November, 2019