Bianchi type -II, VIII & IX Holographic Dark energy Cosmological Models in Brans-Dicke theory of gravitation

K. V. S. Sireesha[1], V.U.M.Rao*[2]
[1]*Department of Engineering Mathematics, GITAM University, Visakhapatnam, India*
[2]*Department of Applied Mathematics, Andhra University, Visakhapatnam, India*
*sireeshakuppili@gmail.com *umrao57@hotmail.com*

The present study deals with spatially homogeneous Bianchi type-II, VIII & IX holographic dark energy cosmological models in Brans and Dicke (1961) theory of gravitation. To get the deterministic models of the Universe, we have used a power law between scalar field and the scale factor of the Universe. It has been found that the anisotropic distribution of dark energy leads to the present accelerated expansion of the Universe. All the models obtained and presented here are expanding, non-rotating and accelerating. Also some important features of the models including look-back time, distance modulus and luminosity distance versus red shift with their significances are discussed.

1. Introduction

It has been believed that the Universe is undergoing a phase of accelerated expansion, which is indicated by such astronomical observations as Type-I a supernovae (SNe) (Riess et al.[2]), cosmic microwave background (CMB) anisotropy (Spergel et al.[3]) and large scale structure (LSS) (Tegmark et al. [4]). This implies that there is a mysterious component in the Universe, which has a large negative pressure called dark energy (DE). The Wilkinson Microwave Anisotropy Probe (WMAP) satellite experiment indicates that the Universe is spatially flat on a large scale and the dark energy, dark matter (DM) and baryon matter in the Universe make up about 73%, 23%, and 4%, respectively.

Recently, there were some studies showing that the equation of state (EoS) of dark energy $w_x = \dfrac{p_x}{\rho_x}$ might evolve from $w_x > -1$ to $w_x < -1$. When these SNe results are combined with WMAP 5-year data the 95% confidence limits on non-evolving EoS are $-1.11 < w_{ox} < -0.86$ (Hinshaw et al. [5]). More scenarios have been proposed as the candidates of DE to explain the accelerated expansion. What fits best with the observational data is the cosmological constant λ (Carroll [6]), which has the EoS $w = -1$ and is called the LCDM model in which the cosmological constant is combined with the cold dark matter. However, it is plagued with some problems i.e., the "cosmic coincidence" problem (Fitch et al. [7]) and the "fine tuning" problem (Peebles and Ratra [8]).

A number of scalar field models have been constructed such as quintessence (Padmanabhan [9]), phantom (Caldwell [10]), K-essence (Chiba et al. [11]), quintom (Feng et al. [12]) and tachyon (Sen [13]). Further, models such as Chaplygin gas (CG) (Kamenshchik et al. [14]) and modified Chaplygin gas (MCG) (Chimento [15]), Wu Y B et al. [16, 17], which attempt to unify DE and DM, are proposed by allowing for a fluid with an EoS that evolves between them. Recently, the most remarkable observational discoveries of distant type-I a supernovae and cosmic microwave back ground have shown that our universe has entered a phase of accelerated expansion in the recent past (Perlmutter et al. [18], Riess et al. [2], Bennett et al. [19]). These observations have made it clear that the current matter-energy density of the Universe is close to its critical value of which 30 % is attributed to relativistic matter including both baryons and dark matter and 70 % to dark energy (Ade et al. [20]).

The cause of sudden transition from the earlier deceleration phase to the recent acceleration phase and the source of accelerated expansion are still unknown. According to the Einstein's theory of general relativity, the cause of such acceleration, one need to introduce a component to the matter distribution of the Universe with a large negative pressure and makes up about three quarters of the total cosmic density.This exotic type of unknown repulsive force is termed as dark energy (DE). Recently many radically different models have been proposed to satisfy the present value of DE. The simplest candidate for DE is the cosmological constant (Λ) with equation of state parameter $\omega = -1$ since it fits the observational data well, but it needs to be extremely fine-tuned to satisfy the current value of DE (Copeland et al. [21]). At present Λ with a dynamical character is preferred over a constant Λ to solve cosmological constant problem especially a time dependent Λ which has decreased slowly from its large initial value to reach its present small value (Overduin et al. [22]).

To further investigate the properties of dark energy, many dynamical dark energy models have been proposed such as quintessence with EoS $\omega > -1$ (Barreiro et al. [23]), phantom with EoS $\omega < -1$ (Caldwell [24]), tachyon (Bagla et al. [25], Sen [13]), Padmanabhan and Choudhury [26], *k*-essence (Armendariz et al. [27]), dilatonic ghost condensate (Gasperini et al. [28]), quartessence (Leon et al. [29]) and so forth. The cosmic viscosity is also an effective quantity as caused mainly by the non-perfect cosmic contents interactions and may play a role as dark energy candidate causing the observed acceleration of the universe (Zimdahl et al. [30], Cataldo et al. [31]).

Holographic dark energy is the nature of DE can also be studied according to some basic quantum gravitational principle. According to this principle (Susskind [32]), the degrees of freedom in a bounded system should be finite and does not scale by it volume but with its boundary era. Here ρ_Λ is the vacuum energy density. Usingthis idea in cosmology we take ρ_Λ as DE density. The holographic principle is considered as another alternativeto the solution of DE problem. This principle was first considered by ’t Hooft [33] in the context of black hole physics.

In the context of dark energy problem though the holographic principle proposes a relation between the holographic dark energy density ρ_Λ and the Hubble parameter H as $\rho_\Lambda = H^2$, it does not contribute to the present accelerated expansion of the Universe. Granda and Olivers [34] have proposed a holographic density of the form $\rho_\Lambda \approx \alpha H^2 + \beta \dot{H}$, where H is the Hubble parameter and α, β are constants, which must satisfy the conditions imposed by the current observational data. They showed that this new model of dark energy represents the accelerated expansion of the Universe and is consistent with the current observational data. Granda and Olivers [35] have also studied the correspondence between the quintessence, tachyon, *k*-essence and dilation dark energy models with this holographic dark energy model in the flat FRW universe. But there is a cosmological view that the universe might have been anisotropic and also inhomogeneous in the very early era and that in the course of its evolution these characteristics might have been wiped out under the action of some processes or mechanism, resulting in an isotropic and homogeneous universe.

Spatially homogeneous and anisotropic cosmological models play a significant role in the description of large scale behavior of the Universe and such models have been widely studied by many authors in search of a relativistic picture of the early universe. Recently, Kiran et al. [36,37] have studied minimally interacting dark energy models in scalar tensor theories. Adhav et al. [38] have discussed interacting dark matter and holographic dark energy in Bianchi type-V universe.

Brans-Dicke [1] theory of gravitation is a natural extension of general relativity which introduces an additional scalar field ϕ besides the metric tensor g_{ij} and dimensionless coupling constant ω. The Brans - Dicke [1] field equations for combined scalar and tensor field are given by

$$G_{ij} = -8\pi\phi^{-1}T_{ij} - \omega\phi^{-2}\left(\phi_{,i}\,\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{'k}\right) - \phi^{-1}\left(\phi_{i,j} - g_{ij}\phi^{'k}_{;k}\right) \quad (1)$$

and $$\varphi_{;k}{}^{,k} = 8\pi(3+2\omega)^{-1}T \quad (2)$$

Where, $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ is an Einstein tensor, R is the scalar curvature, ω and n are constants, T_{ij} is the stress energy tensor of the matter and comma and semicolon denote partial and covariant differentiation, respectively.

Also, we have energy – conservation equation

$$T^{ij}_{;j} = 0 \quad (3)$$

Several aspects of Brans-Dicke cosmology have been extensively investigated by many authors. Rao et al. [39] have studied LRS Bianchi type-I dark energy cosmological model in Brans-Dicke theory of gravitation. Rao and Sireesha [40] have discussed Bianchi types II, VIII & IX string cosmological models with bulk viscosity in Brans-Dicke theory of gravitation. Rao and Sireesha [41] have studied higher dimensional string cosmological model in a scalar-tensor theory of gravitation. Rao and Sireesha [42] have investigated Bianchi type-II, VIII & IX cosmological models with strange quark matter attached to string cloud in Brans-Dicke and General theory of gravitation.

Bianchi type space-times play a vital role in understanding and description of the early stages of evolution of the Universe. In particular, the study of Bianchi types II, VIII & IX universes are important because familiar solutions like FRW universe with positive curvature, the de Sitter universe, and the Taub-Nut solutions correspond Bianchi type II, VIII & IX space- times. Rao et al. [43] have studied Bianchi types II, VIII & IX string cosmological models with bulk viscosity in a theory of gravitation. Rao et al. [44] have investigated Bianchi type-II, VIII and IX dark energy cosmological model in Saez-Ballester theory of gravitation. Rao et al. [45] have discussed perfect fluid cosmological models in a modified theory of gravity.

This paper is outlined as follows. In Sec. 2, we have obtained the Brans - Dicke field equations for Bianchi type-II, VIII and IX metric in the presence of matter and holographic dark energy. In Sec. 3, we obtained the solution of the field equations. We also discuss some of the features of this model including effective EoS and the evolution of energy density between DE and DM. In Sec.4, we discuss some important properties of the model. Some conclusions are presented in the last section.

2. Metric and Energy Momentum Tensor

We consider a spatially homogeneous Bianchi type-II,

VIII & IX metrics of the form

$$ds^2 = dt^2 - R^2\left[d\theta^2 + f^2(\theta)d\phi^2\right] - S^2\left[d\psi + h(\theta)d\phi\right]^2 \tag{4}$$

Where, (θ,φ,ψ) are the Eulerian angles, Rand S are functions of t only.
It represents

Bianchi type - II if $f(\theta)=1$ and $h(\theta)=\theta$

Bianchi type-VIII if $f(\theta)=Cosh\,\theta$ and $h(\theta)=Sinh\,\theta$

Bianchi type - IX if $f(\theta)=Sin\,\theta$ and $h(\theta)=Cos\,\theta$

The energy momentum tensors for matter and the holographic energy are defined as

$$T_{ij} = \rho_m u_i u_j \tag{5}$$

and $$\overline{T}_{ij} = \left(\rho_\Lambda + p_\Lambda\right)u_i u_j - g_{ij}\rho_\Lambda \tag{6}$$

Where ρ_m, ρ_Λ are energy densities of matter and holographic dark energy and p_Λ is the pressure of holographic dark energy.
In a co moving coordinate system, we get

$$T_1^{\ 1} = T_2^{\ 2} = T_3^{\ 3} = 0,\ T_4^{\ 4} = \rho_m \text{ and}$$

$$\overline{T}_1^{\ 1} = \overline{T}_2^{\ 2} = \overline{T}_3^{\ 3} = -p_\Lambda,\ \overline{T}_4^{\ 4} = \rho_\Lambda \tag{7}$$

Where, the quantities ρ_m, ρ_Λ and p_Λ are functions of 't' only.

3. Solutions of Field equations

Now with the help of (5) to (7), the field equations (1) for the metric (4) can be written as

$$\frac{\ddot R}{R}+\frac{\ddot S}{S}+\frac{\dot R\dot S}{RS}+\frac{S^2}{4R^4}+\frac{\omega\dot\phi^2}{2\phi^2}+\frac{\ddot\phi}{\phi}+\frac{\dot R\dot\phi}{R\phi}+\frac{\dot S\dot\phi}{S\phi}=-\frac{8\pi\, p_\Lambda}{\phi} \tag{8}$$

$$2\frac{\ddot R}{R}+\frac{\dot R^2+\delta}{R^2}-\frac{3S^2}{4R^4}+\frac{\omega\dot\phi^2}{2\phi^2}+\frac{\ddot\phi}{\phi}+2\frac{\dot R\dot\phi}{R\phi}=-\frac{8\pi\, p_\Lambda}{\phi} \tag{9}$$

$$2\frac{\dot R\dot S}{RS}+\frac{\dot R^2+\delta}{R^2}-\frac{S^2}{4R^4}-\frac{\omega\dot\phi^2}{2\phi^2}+2\frac{\dot R\dot\phi}{R\phi}+\frac{\dot S\dot\phi}{S\phi}=\frac{8\pi(\rho_m+\rho_\Lambda)}{\phi} \tag{10}$$

$$\left(\frac{\dot S}{S}-\frac{\dot R}{R}\right)\frac{h(\theta)\dot\phi}{\phi}=0 \tag{11}$$

$$\ddot\phi+\dot\phi\left(\frac{2\dot R}{R}+\frac{\dot S}{S}\right)=\frac{8\pi}{(3+2\omega)}(\rho_m+\rho_\Lambda-3p_\Lambda) \tag{12}$$

$$\dot\rho_m+\dot\rho_\Lambda+\left(\frac{2\dot R}{R}+\frac{\dot S}{S}\right)\left(\rho_m+\rho_\Lambda+P_\Lambda\right)=0 \tag{13}$$

Here the overhead dot denotes differentiation with respect to 't'.

When $\delta = 0,-1\ \&+1$ the field equations (8) to (13) correspond to the Bianchi types II, VIII & IX universes, respectively.

Using the transformation $dt = R^2 S dT$, the above field equations (8) to (13) will reduce to

$$\left(\frac{R'}{R}\right)'+\left(\frac{S'}{S}\right)'-\frac{R'}{R}\left(\frac{R'}{R}+\frac{2S'}{S}\right)+\frac{\omega\phi'^2}{2\phi^2}+\frac{\phi''}{\phi}-\frac{R'\phi'}{R\phi}+\frac{S^4}{4}=-\frac{8\pi\, p_\Lambda}{\phi}R^4S^2 \tag{14}$$

$$2\left(\frac{R'}{R}\right)'-\frac{R'}{R}\left(\frac{R'}{R}+\frac{2S'}{S}\right)+\delta R^2S^2+\frac{\omega\phi'^2}{2\phi^2}+\frac{\phi''}{\phi}-\frac{S'\phi'}{S\phi}-\frac{3S^4}{4}=-\frac{8\pi\, p_\Lambda}{\phi}R^4S^2 \tag{15}$$

$$\frac{R'}{R}\left(\frac{R'}{R}+\frac{2S'}{S}\right)+\delta R^2S^2-\frac{\omega\phi'^2}{2\phi^2}+\frac{2R'\phi'}{R\phi}+\frac{S'\phi'}{S\phi}-\frac{S^4}{4}=\frac{8\pi(\rho_m+\rho_\Lambda)}{\phi}R^4S^2 \tag{16}$$

$$\left(\frac{S'}{S}-\frac{R'}{R}\right)\frac{h(\theta)\phi'}{\phi}=0 \tag{17}$$

$$\phi''=\frac{8\pi}{(3+2\omega)}(\rho_m+\rho_\Lambda-3p_\Lambda)R^4S^2 \tag{18}$$

$$\rho'_m+\rho'_\Lambda+\left(\frac{2R'}{R}+\frac{S'}{S}\right)\left(\rho_m+\rho_\Lambda+P_\Lambda\right)=0 \tag{19}$$

Here the overhead dash denotes differentiation with respect to 'T'.

Since we are considering the Bianchi type-II, VIII & IX metrics, we have $h(\theta)=\theta, h(\theta)=\sinh\theta\ \&\ h(\theta)=\cos\theta$ respectively. Therefore, from the equation (17), we will get the following possible cases with $h(\theta)\neq 0$

(1) $\frac{S'}{S}-\frac{R'}{R}=0 \quad and \quad \phi'\neq 0$

(2) $\frac{S'}{S}-\frac{R'}{R}\neq 0 \quad and \quad \phi'=0$

(3) $\frac{S'}{S}-\frac{R'}{R}=0 \quad and \quad \phi'=0$

From the above three possibilities, we will consider only the 1st possibility since in other two cases we will get cosmological models in general relativity.

3.1 Cosmological models in Brans-Dicke theory

We will get cosmological models in Brans-Dicke theory only in case of $\frac{S'}{S}-\frac{R'}{R}=0 \quad and \quad \phi'\neq 0$.

If $\frac{S'}{S}-\frac{R'}{R}=0$, we get $R=S+c$.

Without loss of generality by taking the constant of integration $c=0$, we get $R=S$ (20)

The field equations (14) to (16) are only five independent equations with seven unknowns $R, S, \rho_m, \rho_\Lambda, p_\Lambda$ & ϕ, which are functions of 'T'. Since these equations are non-linear in nature, in order to get a deterministic solution we take the following plausible physical condition:

The relation between the scalar field ϕ and the scale factor of the universe $a(t)$ given by

$$\phi=\phi_0 a^m \tag{21}$$

where ϕ_0 and $m>0$ are constants.

From equations (14) – (16), (18) & (19), we get

$$2\frac{R''}{R}+k_1\frac{R'^2}{R^2}=\frac{(4\delta-1)}{2(3-\omega m)}R^4 \tag{22}$$

Where, $k_1=\frac{2\omega m-\omega m^2-12}{(3-\omega m)}$.

The continuity equation can be obtained as

$$\rho'_m+\rho'_\Lambda+\left(\frac{2R'}{R}+\frac{S'}{S}\right)\left(\rho_m+\rho_\Lambda+p_\Lambda\right)=0 \tag{23}$$

The continuity equation of the matter is

$$\rho'_m+\left(\frac{2R'}{R}+\frac{S'}{S}\right)\rho_m=0 \tag{24}$$

The continuity equation of the holographic dark energy is

$$\rho'_\Lambda+\left(\frac{2R'}{R}+\frac{S'}{S}\right)\left(\rho_\Lambda+p_\Lambda\right)=0 \tag{25}$$

The barotropic equation of state

$$p_\Lambda=\omega_\Lambda\rho_\Lambda \tag{26}$$

3.2 Bianchi type-II ($\delta=0$) cosmological model

If $\delta=0$, the equation (22) can be written as

$$2\frac{R''}{R}+k_1\frac{R'^2}{R^2}=\frac{1}{2(\omega m-3)}R^4 \tag{27}$$

From equation (27), we get

$$R=[2(k_2T+k_3)]^{-1/2} \tag{28}$$

Where, $k_2=\frac{1}{2(3-\omega m)(k_1+6)}$ and k_3 is an integrating constant.

From equations (20) & (28), we get

$$S=[2(k_2T+k_3)]^{-1/2} \tag{29}$$

From equations (21), (28) & (29), we get

$$\phi=\phi_0[2(k_2T+k_3)]^{-m/2} \tag{30}$$

The holographic dark energy density are given by

$$\rho_\Lambda=\frac{2}{\alpha-\beta}\left(H'+\frac{3\alpha}{2}H^2\right) \tag{31}$$

Where, H is the Hubble parameter, α and β are constants which must satisfy the restrictions imposed by the current observational data.

From equations (31), (28) & (29), we get the holographic dark energy density

$$8\pi\rho_\Lambda=\frac{8\pi(4+3\alpha)}{4(\alpha-\beta)}\left(\frac{k_2^2}{(k_2T+k_3)^2}\right) \tag{32}$$

From equations (16), (28), (29), (30) & (32), we get the matter energy density

$$8\pi\rho_m=\frac{\{[12(m+1)-2\omega m^2]k_2^2-1\}\phi_0}{4[2(k_2T+k_3)]^{\frac{m-2}{2}}}-\frac{8\pi(4+3\alpha)}{4(\alpha-\beta)}\left(\frac{k_2^2}{(k_2T+k_3)^2}\right) \tag{33}$$

From equations (15), (28), (29) & (30), we get the pressure of holographic dark energy

$$8\pi p_\Lambda=\frac{\{3-[2(\omega+2)m^2+4m+4]k_2^2\}\phi_0}{4[2(k_2T+k_3)]^{\frac{m-2}{2}}} \tag{34}$$

From equations (26),(32) & (34), we get

$$w_\Lambda=\frac{\{3-[2(\omega+2)m^2+4m+4]k_2^2\}(\alpha-\beta)\phi_0}{4k_2^2(4+3\alpha)[2(k_2T+k_3)]^{\frac{m-6}{2}}} \tag{35}$$

The coincident parameter is

$$r=\frac{\rho_\Lambda}{\rho_m}=\frac{\frac{8\pi(4+3\alpha)}{4(\alpha-\beta)}\left(\frac{k_2^{\,2}}{(k_2T+k_3)^2}\right)}{\left\{\begin{array}{l}\frac{\{[12(m+1)-2\omega m^2]k_2^{\,2}-1\}\phi_0}{4[2(k_2T+k_3)]^{\frac{m-2}{2}}}-\\ \frac{8\pi(4+3\alpha)}{4(\alpha-\beta)}\left(\frac{k_2^{\,2}}{(k_2T+k_3)^2}\right)\end{array}\right\}} \tag{36}$$

The metric (4) in this case can be written as

$$ds^2=\left\{\begin{array}{l}[2(k_2T+k_3)]^{-3/2}\,dT^2-[2(k_2T+k_3)]^{-1}\\ \left(d\theta^2+d\phi^2\right)-[2(k_2T+k_3)]^{-1}\left(d\varphi+\theta\,d\phi\right)^2\end{array}\right\} \tag{37}$$

Thus (37) together with (32) to (35) constitutes a Bianchi type-II holographic dark energy cosmological model in Brans-Dicke [1] scalar tensor theory of gravitation.

3.3 Bianchi type-VIII ($\delta=-1$) cosmological model

If, $\delta=-1$, the equation (22) can be written as

$$2\frac{R''}{R}+k_1\frac{R'^2}{R^2}=\frac{5}{2(\omega m-3)}R^4 \tag{38}$$

From equation (38), we get

$$R=[2(k_2T+k_3)]^{-1/2} \tag{39}$$

Where, $k_2=\frac{5}{2(3-\omega m)(k_1+6)}$ and k_3 is an integrating constant.

From equations (20) & (39), we get

$$S=[2(k_2T+k_3)]^{-1/2} \tag{40}$$

From equations (21), (39) & (40), we get

$$\phi=\phi_0\,[2(k_2T+k_3)]^{-m/2} \tag{41}$$

The holographic dark energy density are given by

$$\rho_\Lambda=\frac{2}{\alpha-\beta}\left(H'+\frac{3\alpha}{2}H^2\right) \tag{42}$$

Where, H is the Hubble parameter.

From equations (42), (39) & (40), we get the holographic dark energy density

$$8\pi\rho_\Lambda=\frac{8\pi(4+3\alpha)}{4(\alpha-\beta)}\left(\frac{k_2^{\,2}}{(k_2T+k_3)^2}\right) \tag{43}$$

From equations (16), (39),(40),(41) & (43), we get the matter energy density

$$8\pi\rho_m=\frac{\{[12(m+1)-2\omega m^2]k_2^{\,2}-5\}\phi_0}{4[2(k_2T+k_3)]^{\frac{m-2}{2}}}-\frac{8\pi(4+3\alpha)}{4(\alpha-\beta)}\left(\frac{k_2^{\,2}}{(k_2T+k_3)^2}\right) \tag{44}$$

From equations (15), (39), (40) & (41), we get the pressure of holographic dark energy

$$8\pi p_\Lambda=\frac{\{7-[2(\omega+2)m^2+4m+4]k_2^{\,2}\}\phi_0}{4[2(k_2T+k_3)]^{\frac{m-2}{2}}} \tag{45}$$

From equations (26),(43) & (45), we get

$$w_\Lambda=\frac{\{7-[2(\omega+2)m^2+4m+4]k_2^{\,2}\}(\alpha-\beta)\,\phi_0}{4k_2^{\,2}(4+3\alpha)[2(k_2T+k_3)]^{\frac{m-6}{2}}} \tag{46}$$

The coincident parameter is

$$r=\frac{\rho_\Lambda}{\rho_m}=\frac{\frac{8\pi(4+3\alpha)}{4(\alpha-\beta)}\left(\frac{k_2^{\,2}}{(k_2T+k_3)^2}\right)}{\left\{\begin{array}{l}\frac{\{[12(m+1)-2\omega m^2]k_2^{\,2}-5\}\phi_0}{4[2(k_2T+k_3)]^{\frac{m-2}{2}}}-\\ \frac{8\pi(4+3\alpha)}{4(\alpha-\beta)}\left(\frac{k_2^{\,2}}{(k_2T+k_3)^2}\right)\end{array}\right\}} \tag{47}$$

The metric in Eqn. (4), in this case can be written as

$$ds^2=\left\{\begin{array}{l}[2(k_2T+k_3)]^{-3/2}\,dT^2-[2(k_2T+k_3)]^{-1}\\ \left(d\theta^2+\cosh^2\theta\,d\phi^2\right)-[2(k_2T+k_3)]^{-1}\left(d\varphi+\sinh\theta\,d\phi\right)^2\end{array}\right\} \tag{48}$$

Thus Eqn. (48) together with Eqns. (43) to (47) constitutes a Bianchi type-VIII holographic dark energy cosmological model in Brans-Dicke [1] scalar tensor theory of gravitation.

3.4 Bianchi type-IX ($\delta=1$) cosmological model

If, $\delta=1$, the equation (22) can be written as

$$2\frac{R''}{R}+k_1\frac{R'^2}{R^2}=\frac{3}{2(3-\omega m)}R^4 \tag{49}$$

From equation (49), we get

$$R=[2(k_2T+k_3)]^{-1/2} \tag{50}$$

Where, $k_2 = \dfrac{3}{2(\omega m - 3)(k_1 + 6)}$ and k_3 is an integrating constant.

From equations (20) & (50), we get

$$S = [2(k_2 T + k_3)]^{-1/2} \tag{51}$$

From equations (21), (50) & (51), we get

$$\phi = \phi_0 [2(k_2 T + k_3)]^{-m/2} \tag{52}$$

The holographic dark energy density are given by

$$\rho_\Lambda = \frac{2}{\alpha - \beta}\left(H' + \frac{3\alpha}{2} H^2\right) \tag{53}$$

Where, H is the Hubble parameter.

From equations (53), (50) & (51), we get the holographic dark energy density

$$8\pi\rho_\Lambda = \frac{8\pi(4+3\alpha)}{4(\alpha-\beta)}\left(\frac{k_2^{\,2}}{(k_2 T + k_3)^2}\right) \tag{54}$$

From equations (16), (50), (51),(52) & (54), we get the matter energy density

$$8\pi\rho_m = \left\{ \begin{array}{c} \dfrac{\{[12(m+1) - 2\omega m^2]k_2^{\,2} + 3\}\phi_0}{4[2(k_2 T + k_3)]^{\frac{m-2}{2}}} - \\ \dfrac{8\pi(4+3\alpha)}{4(\alpha-\beta)}\left(\dfrac{k_2^{\,2}}{(k_2 T + k_3)^2}\right) \end{array} \right\} \tag{55}$$

From equations (15), (50), (51) & (52), we get the pressure of holographic dark energy

$$8\pi\, p_\Lambda = \frac{\{-1 - [2(\omega+2)m^2 + 4m + 4]k_2^{\,2}\}\phi_0}{4[2(k_2 T + k_3)]^{\frac{m-2}{2}}} \tag{56}$$

From equations (26), (54) & (56), we get

$$w_\Lambda = \frac{\{1 + [2(\omega+2)m^2 + 4m + 4]k_2^{\,2}\}(\beta - \alpha)\phi_0}{4k_2^{\,2}(4+3\alpha)[2(k_2 T + k_3)]^{\frac{m-6}{2}}} \tag{57}$$

The coincident parameter is

$$r = \frac{\rho_\Lambda}{\rho_m} = \frac{\dfrac{8\pi(4+3\alpha)}{4(\alpha-\beta)}\left(\dfrac{k_2^{\,2}}{(k_2 T + k_3)^2}\right)}{\left\{ \begin{array}{c} \dfrac{\{[12(m+1) - 2\omega m^2]k_2^{\,2} + 3\}\phi_0}{4[2(k_2 T + k_3)]^{\frac{m-2}{2}}} - \\ \dfrac{8\pi(4+3\alpha)}{4(\alpha-\beta)}\left(\dfrac{k_2^{\,2}}{(k_2 T + k_3)^2}\right) \end{array} \right\}} \tag{58}$$

The metric (4), in this case can be written as

$$ds^2 = \left\{ \begin{array}{l} [2(k_2 T + k_3)]^{-3/2} dT^2 - [2(k_2 T + k_3)]^{-1} \\ \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) - [2(k_2 T + k_3)]^{-1}\left(d\varphi + \cos\theta\, d\phi\right)^2 \end{array} \right\} \tag{59}$$

Thus (59) together with (54) to (58) constitutes a Bianchi type-IX holographic dark energy cosmological model in Brans-Dicke [1] scalar tensor theory of gravitation.

4. Some other important properties of the models

The spatial volume for the models is

$$V = (-g)^{\frac{1}{2}} = [2(k_2 T + k_3)]^{-3/2} f(\theta) \tag{60}$$

Where,

$k_2 = \dfrac{1}{2(3 - \omega m)(k_1 + 6)}$ & $f(\theta) = \theta$, $k_2 = \dfrac{5}{2(3 - \omega m)(k_1 + 6)}$ & $f(\theta)$

and $k_2 = \dfrac{3}{2(3 - \omega m)(k_1 + 6)}$ & $f(\theta) = Sin\theta$

for Bianchi type-II,VIII & IX respectively.

The average scale factor for the model is

$$a(t) = V^{\frac{1}{3}} = [2(k_2 T + k_3)]^{-1/2}[f(\theta)]^{1/3} \tag{61}$$

The expression for expansion scalar θ calculated for the flow vector u^i is given by

$$\theta = u^i{}_{,i} = \frac{-3}{2}\frac{k_2}{(k_2 T + k_3)} \tag{62}$$

and the shear scalar σ is given by

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij} = \frac{9}{8}\frac{k_2^{\,2}}{(k_2 T + k_3)^2} \tag{63}$$

The deceleration parameter q is given by

$$q = (-3\theta^{-2})(\theta_{,i}u^i + \frac{1}{3}\theta^2) = -3 \tag{64}$$

The deceleration parameter appears with negative sign implies accelerating expansion of the universe, which is consistent with the present day observations.

The Hubble's parameter H is given by

$$H = \frac{-1}{2}\frac{k_2}{(k_2 T + k_3)} \tag{65}$$

The mean anisotropy parameter A_m is given by

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 = 0$$ Where,

$$\Delta H_i = H_i - H \quad (i = 1,2,3) \tag{66}$$

Look-back time and red shift: the look-back time, $\Delta t = t_0 - t(z)$ is the difference between the age of the Universe at present time (z=0) and the age of the Universe when a particular light ray at red shift z, the expansion scalar of the universe $a(t_z)$ is related to a_0 by $1 + z = \frac{a_0}{a}$, where a_0 is the present scale factor. Therefore from (4.2), we get

$$1 + z = \frac{a_0}{a} = \left(\frac{k_2 T_0 + k_3}{k_2 T + k_3}\right)^{\frac{-1}{2}} \tag{67}$$

This equation can also be expressed as

$$H_0 \Delta T = 1 - (1+z)^2 \tag{68}$$

Where, H_0 is the Hubble's constant.

Luminosity distance:
Luminosity distance is defined as the distance which will preserve the validity of the inverse law for the fall of intensity and, is given by

$$d_L = r_1 (1+z) a_0 \tag{69}$$

Where, r_1 is the radial coordinate distance of the object at light emission and, is given by

$$r_1 = \int_{T0}^{T} \frac{1}{a} dT$$

$$= \frac{2\sqrt{2}}{3k_2}(k_2 T_0 + k_3)^{\frac{3}{2}}\left[1 - (1+z)^3\right] \tag{70}$$

From equations (69) and (70), we get
The luminosity distance

$$d_L = \frac{2\sqrt{2}}{3k_2} a_0 (1+z)(k_2 T_0 + k_3)^{\frac{3}{2}}\left[1 - (1+z)^3\right] \tag{71}$$

From equations (70) and (71), we get
The distance modulus

$$D(z) = 5\log\left\{\frac{2\sqrt{2}}{3k_2} a_0 (1+z)(k_2 T_0 + k_3)^{\frac{3}{2}}\left[1 - (1+z)^3\right]\right\} + 25 \tag{72}$$

The tensor of rotation $w_{ij} = u_{i,j} - u_{j,i}$ is identically zero and hence this universe is non-rotational.

5. Discussion and Conclusions

In this paper, we have presented spatially homogeneous Bianchi type - II, VIII & IX holographic dark energy cosmological models in Brans-Dicke [1] scalar tensor theory of gravitation.
The following are the observations and conclusions.

- The models are always isotropic and have singularity at $T = \frac{-k_3}{k_2}$.
- The volume decreases with the increase of time i.e., as $T \to \infty$, the spatial volume vanishes.
- At $T = \frac{-k_3}{k_2}$, the expansion scalar θ, shear scalar σ and the Hubble parameter H decreases with the increase of time.
- From (66), one can observe that $A_m = 0$ and this indicates that these universes always expand isotropically.
- For all the three models, the energy density, the pressure and the coincident parameter of holographic dark energy will decreases with the increase of time' T '.
- The deceleration parameter appears with negative sign implies accelerating expansion of the universe and hence it represents present universe.
- We have obtained expressions for look-back time ΔT, distance modulus $D(z)$ and luminosity distance d_L versus red shift and discussed their significance.
- All the models presented here are isotropic, non-rotating, shearing and also accelerating.

References

[1] Brans.C.H.and Dicke.R.H., Phys.Rev.124, 925 (1961).
[2] Riess A G, Filippenko Al V, Challis P, et al., Astron. J., 116,1009 (1998).
[3] Spergel D N, Verde L, Peiris H V, et al., Astrophys. J. Suppl., 148, 175 (2003).

[4] Tegmark M, Strauss M, Blanton M, et al., Phys. Rev. D., 69, 103501 (2004).
[5] Hinshaw G, Weiland J L, Hill R S, et al., Astrophys. J. Suppl., 180, 225 (2009).
[6] Carroll S M., Living Rev. Rel., 4, 1(2001).
[7] Fitch V L, Marlow D R, Dementi M A E. Critical, Princeton University Press, (1997).
[8] Peebles P J E, Ratra B., Rev. Mod. Phys., 75, 559 (2003).
[9] Padmanabhan T., Phys. Rep., 380, 235 (2003).
[10] Caldwell R R., Phys.Lett. B., 545, 23(2002).
[11] Chiba T, Okabe T, Yamaguchi M.L, Phys. Rev. D., 62, 023511 (2000).
[12] Feng B, Wang X L, Zhang X M., Phys.Lett. B., 607, 35 (2005).
[13] Sen A. ,J. High Energy Phys., 0207, 065 (2002).
[14] Kamenshchik A, Moschella U, Pasquier V., Phys.Lett. B., 511, 265 (2001).
[15] Chimento L P., Phys. Rev. D., 69,123517 (2004).
[16] Wu Y B, Li S, Lu J B, et al.: ModernPhys.Lett. A, 22(11), 783 (2007).
[17] Wu Y B, He J, Wang C, et al., Modern Phys.Lett. A, 24(1), 63 (2009).
[18] Perlmutter, S., et al., Nature (London) 391, 51 (1998).
[19] Bennett, C.L., et al., Astrophys. J. Suppl. Ser. 148, 1 (2003).
[20] Ade, P.A.R., et al., arXiv:1303.5076 (2013).
[21] Copeland, E.J., Sami, M., Tsujikawa, S., Int. J. Mod. Phys. D 15, 1753(2006).
[22] Overduin, J.M., et al., Phys. Rev. D 58, 043506 (1998).
[23] Barreiro, T., et al., Phys. Rev. D 61, 127301 (2000).
[24] Caldwell, R.R., Phys. Rev. Lett. 91, 071301 (2003).
[25] Bagla, J.S., et al.: Phys. Rev. D 67, 063504 (2003).
[26] Padmanabhan, T., Choudhury, T.R., Phys. Rev. D 66, 081301 (2002).
[27] Armendariz, C., et al., Phys. Rev. D 63, 103510 (2001).
[28] Gasperini, M., et al., Phys. Rev. D **65**, 023508 (2002).
[29] Leon, G., et al., Phys. Lett. B 693, 1 (2010).
[30] Zimdahl, W., et al., Phys. Rev. D 64, 063501 (2001).
[31] Cataldo, M., et al., Phys. Lett. B 619, 5 (2005).
[32] Susskind, L, J. Math. Phys. 36, 6377(1995).
[33] ‘t Hooft, G., gr-qc/9310026 (2009).
[34] Granda, L. N., Oliveros, A., Phys. Lett. B 669, 275(2008).
[35] Granda, L.N., Oliveros, A., Phys. Lett. B671, 199 (2009).
[36] Kiran,M., Reddy, D.R.K., Rao, V.U.M., Astrophys.Space. Sci., 354,577 (2014).
[37] Kiran,M., Reddy, D.R.K., Rao, V.U.M., Astrophys.Space. Sci., 356,407(2015).
[38] Adhav,K.S., Munde,S.L.,Tayade, G.B., Bokey V.D, Astrophys. Space Sci.359,24 (2015).
[39] Rao, V. U. M., VijayaSanthi, M., Vinutha, T. and SreedeviKumari,G., Int. J.Theor. Phys.51, 3303 (2012).
[40] Rao, V.U.M., Sireesha, K.V.S., Int. J. Theor. Phys.51, 3013 (2012).
[41] Rao, V.U.M., Sireesha, K.V.S., Eur. Phys. J. Plus 127, 33 (2012).
[42] Rao, V.U.M., Sireesha, K.V.S., Int. J. Theor. Phys.52, 1240 (2013).
[43] Rao, V.U.M., Sireesha, K.V.S., VijayaSanthi, M., ISRN Math. Phys., DOI: 10.5402/2012/341612 (2012).
[44] Rao, V.U.M., Sireesha, K.V.S., Suneetha, P., African Rev. of Phys.,7, 0054(2012).
[45] Rao, V.U.M., Sireesha, K.V.S., Papa Rao, D.Ch., Eur. Phys. J. Plus, 129, 17 (2014).

Received: 16 May, 2016

Accepted (Revised Version): 30 June, 2019