

Spectrum of Schrödinger Equation with H.L.C. Potential in Non-commutative Two-dimensional Real Space

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This paper describes the Schrödinger equation for Harmonic-Linear and Colombian H.L.C. potential that was solved by applying the Boopp's shift method to first order in the non-commutativity parameter θ in non-commutative two-dimensional spaces. The modified energy levels at the fundamental state and first excited states are obtained by applying perturbation theory. The non-commutative Hamiltonian is constructed and the new spectrum was obtained which was produced from two modes of interactions: the first one is the effect of an external magnetic field while the second is spin-orbital interaction. The degenerated states are determined for a given orbital quantum number l .

1. Introduction

In the last century, the modern physics has known two major revolutions: the standard model and general relativity. Although, some problems such as the divergences in quantum field theory and the unification of gravitation with standard model remained to be unsolved. Recently, the relativistic Dirac and non-relativistic Schrödinger equations played a crucial role in describing many quantum phenomena in gauge theory, standard model and quantum mechanics at high and low energies, respectively. Furthermore, the exact solution for Colombian and harmonic oscillator potentials in an arbitrary number of spatial dimensions would be interesting since they represent typical models in quantum mechanics and Dirac theory (e.g., in two- and three-dimensional spaces) [1-18]. A new concept of space-time, known as non-commutative space-time, represents some hope to obtain a new and profound interpretation at Planck's scales for solving the divergence problem. The non-commutativity is introduced in many ways. The simplest approach is when the position and momentum operators obeys the Heisenberg commutation relation, which is similar to quantize space-time coordinates, when $\hat{x}_\alpha \hat{x}_\beta \neq \hat{x}_\beta \hat{x}_\alpha$ and

$\Delta x^\alpha \Delta x^\beta \geq |\theta^{\alpha\beta}|$. The formalism of star product,

Boopp's shift method and the Seiberg-Witten map play a fundamental role in this new theory. The rich mathematical structure of the non-commutative theory gives hope to get a better understanding of physical phenomena at small dis-

tances and thus gives further hope to solve the above mentioned problems. The non-commutative space satisfies a new mathematical product, which replaces the old ordinary product, is known as the star product, denoted by $(f(x)*g(x))$ [19-32] and defined as

$$f(x)*g(x) = f(x)g(x) - \frac{i}{2}\theta^{ij}\partial_i f(x)\partial_j g(x) \quad (1)$$

The real parameters θ^{ij} are an anti-symmetric real matrix of dimension length-square. In the particular case, if applying the above equation on the operator's position, one gets the important commutation relation

$$\left[x_i^*, x_j^* \right] = i\theta_{ij} \quad (2)$$

Throughout this paper the natural units ($c = \hbar = 1$) and $\mu = \frac{1}{2}$ are employed. The Boopp's shift method will be used instead of solving the non commutative Schrödinger equation by using star product procedure [19,33,34]

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij} \quad \text{and} \quad [\hat{p}_i, \hat{p}_j] = 0 \quad (3)$$

It is clear that non-commutativity concerns only special operators in the non-relativistic Schrödinger equation. The star product is replaced by the usual product together with a Boopp's shift given as

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$$\begin{aligned}\hat{x}_i &= x_i - \frac{\theta^{ij}}{2} p_j, \quad \hat{p}_i = p_i \\ [x_i, x_j] &= 0, \quad [p_i, p_j] = 0 \\ \dot{x}_i &= [x_i, H] \text{ and } \dot{p}_i = [p_i, H]\end{aligned}\quad (4)$$

Where, $\left(\dot{x}_i, \dot{p}_i\right)$ and H represent the dynamical canonical equation and the Hamiltonian operator of the quantum system.

In this article, we want to study the H.L.C. potential in non-commutative two-dimensional space (NC 2D), the physical importance of this study appeared in nuclear physics [9]. The article is organized as follows. In Sec. 2, we briefly review the Boopp's shift method. In Sec. 3, we briefly review Hydrogen atom in two-dimensional space with H.L.C. potential. In Sec. 4, we derive the deformed Hamiltonian of the Hydrogen atom with H.L.C. potential in NC 2D real space. We solve this equation by applying perturbation theory to obtain the NC modification of the energy levels. Finally, in Sec. 5, we draw our conclusions.

2. Mathematical and Physics Formalism of Boopp's Shift Method

In this section, we shall recall the Boopp's shift method. This method is a consequence of star product between the operator of potential and the wave function in NC space $\hat{V}(\hat{x})$ and $\hat{\Psi}(\hat{x})$, respectively. We replace the star product with the usual product together with following translation [19,33-35]

$$\hat{V}(\hat{x}) * \hat{\Psi}(\hat{x}) \rightarrow V(\hat{x}) \Psi(x) \quad (5)$$

Where, $\hat{x}$ is defined by Eqn. (4). Based on [34-36], the new commutator $[\hat{x}, \hat{y}]$ and the operator $\hat{r}^2$ in NC 2D space are given by

$$\begin{cases} [\hat{x}, \hat{y}] = i\theta \\ \hat{r}^2 = r^2 - \theta L_z \end{cases} \quad (6)$$

Here $(\theta \equiv \theta^{12})$. Now, based on Eqn. (6), we can obtain after a straightforward calculation the following three operators to be used in Sect. 4 so as to apply Boopp's shift method on the modified Schrödinger equation as

$$\begin{cases} \hat{r} = r - \frac{\theta}{2r} L_z \\ \hat{r}^2 = r^2 - \theta L_z \\ \frac{1}{\hat{r}} = \frac{1}{r} + \frac{\theta}{2r^3} L_z \end{cases} \quad (7)$$

3. The H.L.C. Potential in Ordinary Two-dimensional Space

We present the Schrödinger equation with H.L.C. potential, which depend only on the distance r in ordinary two-dimensional space. The best mathematical formulation is to work on the polar coordinate $\vec{r}(r, \varphi)$, in which we have both the usual Schrödinger equation and the H.L.C. potential, respectively as

$$\begin{aligned} \left(-\frac{\Delta}{2m_0} + V(r)\right) \Psi(\vec{r}) &= E \Psi(\vec{r}) \\ V(r) &= ar + br^2 + \frac{c}{r} \end{aligned} \quad (8)$$

Where, m_0 is the mass, Δ is the Laplacien in polar coordinates, a, b and c are arbitrary 3-real numbers representing, electric field, $\frac{1}{2}m\omega^2$ and $(-Z\|e\|)$, respectively, E the energy, and the wave function $\Psi(\vec{r})$ written as

$$\Psi(\vec{r}) = \frac{R_m(r)}{\sqrt{r}} \Phi(\varphi) \quad (9)$$

Where, $\Phi(\varphi)$ and $R_m(r)$ are the angler and radial functions, respectively. The complete (wave function and energies) corresponding to stationary state, first excited starts and p^{th} order excited states are, respectively [9], as

$$\begin{cases} \Psi^{(0)}(\vec{r}) = a_0 r^{\delta-\frac{1}{2}} \exp\left(-\frac{ar+br^2}{2\sqrt{b}}\right) \exp(\pm im\varphi) & \text{and } E_0 = 2\sqrt{b}(1+m) \\ \Psi^{(1)}(\vec{r}) = (a_0 + a_1 r) r^{\delta-\frac{1}{2}} \exp\left(-\frac{ar+br^2}{2\sqrt{b}}\right) & \text{and } E_1 = 2\sqrt{b}(2+m) \\ \Psi^{(p)}(\vec{r}) = (a_0 + a_1 r + \dots + a_p r^p) r^{\delta-\frac{1}{2}} \exp\left(-\frac{ar+br^2}{2\sqrt{b}}\right) & \text{and } E_p = 2\sqrt{b}(1+m+p) \end{cases} \quad (10)$$

Where, $\delta = m + \frac{1}{2}$.

4. H.L.C. Potential in Non-commutative Two Dimensional Real Spaces

The Schrödinger equation in the NC 2D space can be determined from the relation postulated as follows

$$\left(-\frac{1}{2m_0}\Delta + \hat{V}(\hat{r})\right) * \hat{\Psi}(\hat{r}) = E_{NC} \hat{\Psi}(\hat{r}) \quad (11)$$

Where, $\hat{\Psi}(\hat{r})$ is the complete wave function and $\hat{V}(\hat{r})$ is the operator potential and $E_{p1} \equiv 2\pi m\theta A$ is the non-commutative eigenvalues of the energy associated with H.L.C. potential. We shall begin by applying Boopp's shift method to above equation, the deformed Hamiltonian operator H_{hlc} associated to H.L.C. potential in NC 2D space, which is determined by the following equation

$$H_{hlc} = -\frac{\Delta}{2m_0} + V_{hlc}(\hat{r}) \quad (12)$$

Where, the deformed operator of H.L.C. potential $V(\hat{r})$ is given by

$$V_{hlc}(\hat{r}) = a\hat{r} + b\hat{r}^2 + \frac{c}{\hat{r}} \quad (13)$$

Then, Boopp's shift methods imply the following Schrödinger equation in NC 2D space

$$\left(-\frac{1}{2m_0}\Delta + V_{hlc}(\hat{r})\right) \Psi(\hat{r}) = E_{NC} \Psi(\hat{r}) \quad (14)$$

Now, it is clear from Eqn. (7) that we can deduce different terms of $V_{hlc}(\hat{r})$ as

$$\begin{aligned} a\hat{r} &= ar - \frac{\theta a}{2r} L_z \\ b\hat{r}^2 &= br^2 \\ \frac{c}{\hat{r}} &= \frac{c}{r} + \frac{\theta c}{2r^3} L_z \end{aligned} \quad (15)$$

Which, allows us writing the potential $V_{hlc}(\hat{r})$ in the following form

$$V_{hlc}(\hat{r}) = V(r) + H_{hlc-p}(r) \quad (16)$$

Where, $V(r)$ is the ordinary potential, the additive new term $H_{hlc-p}(r)$ takes the form

$$H_{hlc-p}(r) = \theta \left(-\frac{a}{2r} - b + \frac{c}{2r^3}\right) L_z \quad (17)$$

The obtained modified potential $H_{hlc-p}(r)$ is proportional to smallness parameter θ and then we can apply the perturbation theory to obtain the modification to the energy levels E_{hlc-NC}

$$E_{hlc-NC} = \theta m \int \Psi_m^{(p)*}(\hat{r}) \left(-\frac{a}{2r} - b + \frac{c}{2r^3}\right) \Psi_m^{(p)}(\hat{r}) ds \quad (18)$$

Where, $ds = r dr d\phi$, and m is the eigenvalue of L_z . If we apply the generalized Eqn. (23) to the fundamental state, the NC modification of the energy levels, in the first order of θ , will be determined by using Eqns.(10), (17) and (18) as

$$E_{0m} = 2\pi m\theta(A + B + C) \quad (19)$$

Where, the 3 factors A , B and C are given by

$$\begin{cases} A = -\frac{a}{2} a_0^2 \int_0^{+\infty} r^{2\delta-1} \exp\left(-\sqrt{b}r^2 - \frac{a}{\sqrt{b}}r\right) dr \\ B = -ba_0^2 \int_0^{+\infty} r^{2\delta+1-1} \exp\left(-\sqrt{b}r^2 - \frac{a}{\sqrt{b}}r\right) dr \\ C = \frac{c}{2} a_0^2 \int_0^{+\infty} r^{2\delta-2-1} \exp\left(-\sqrt{b}r^2 - \frac{a}{\sqrt{b}}r\right) dr \end{cases} \quad (20)$$

Now, using the special integral [35]

$$\int_0^{+\infty} x^{\nu-1} \exp(-\beta x^2 - \gamma x) dx = (2\beta)^{-\frac{\nu}{2}} \Gamma(\nu) \exp\left(\frac{\gamma^2}{8\beta}\right) D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right) \quad (21)$$

Where, $D_{-\nu}$ are the parabolic cylindrical functions and a condition, $(\text{Re } l(\beta) > 0 \text{ and } \text{Re } l(\nu) > 0)$. This allows us to obtain A , B and C as follows

$$\begin{cases} A = -\frac{a}{2} a_0^2 \left(2\sqrt{b}\right)^{-\frac{2\delta}{2}} \Gamma(2\delta) D_{-2\delta}\left(\frac{a}{\sqrt{2b^{\frac{3}{4}}}}\right) \\ B = -ba_0^2 \left(2\sqrt{b}\right)^{-\frac{2\delta+1}{2}} \Gamma(2\delta) D_{-(2\delta+1)}\left(\frac{a}{\sqrt{2b^{\frac{3}{4}}}}\right) \\ C = \frac{c}{2} a_0^2 \left(2\sqrt{b}\right)^{-\frac{2\delta-2}{2}} \Gamma(2\delta) D_{-(2\delta-2)}\left(\frac{a}{\sqrt{2b^{\frac{3}{4}}}}\right) \end{cases} \quad (22)$$

After a simple calculation, we obtain E_{0m} as

$$E_{0m} = m\theta F(a, b, c) \quad (23)$$

Where, $F(a, b, c)$ is given by

$$F(a, b, c) = 2\pi \begin{pmatrix} -\frac{a}{2} a_0^2 (2\sqrt{b})^{\frac{2\delta}{2}} \Gamma(2\delta) D_{-2\delta} \left(\frac{a}{\sqrt{2b^{\frac{3}{2}}}} \right) \\ -ba_0^2 (2\sqrt{b})^{\frac{2\delta+1}{2}} \Gamma(2\delta) D_{-(2\delta+1)} \left(\frac{a}{\sqrt{2b^{\frac{3}{2}}}} \right) \\ \frac{c}{2} a_0^2 (2\sqrt{b})^{\frac{2\delta-2}{2}} \Gamma(2\delta) D_{-(2\delta-2)} \left(\frac{a}{\sqrt{2b^{\frac{3}{2}}}} \right) \end{pmatrix} \quad (24)$$

Furthermore, the non-commutative modification of the energy levels E_{1m} , in the first order of θ , corresponding to the first excited states, and determined by using Eqns. (10), (17) and (18) as

$$E_{1m} = 2\pi m \theta \int_0^{+\infty} \left[\begin{pmatrix} A_1 r^{2\delta} + A_2 r^{2\delta-1} + A_3 r^{2\delta-3} + \\ A_4 r^{1+2\delta} + A_5 r^{2\delta-2} + A_6 r^{2+2\delta} \end{pmatrix} \right] \exp\left(-\frac{ar+br^2}{2\sqrt{b}}\right) dr \quad (25)$$

If we introduce the notation

$$A_1 = aa_1 a_0 - \mathbf{ba}_0^2, \quad A_2 = \frac{aa_0^2 + ca_1^2}{2}, \quad A_3 = \frac{ca_0^2}{2}, \\ A_4 = \frac{ca_0^2}{2}, \quad A_5 = \frac{aa_1^2}{2} - 2\mathbf{ba}_1 \mathbf{a}_0 \quad \text{and} \quad A_6 = -\mathbf{ba}_1^2 \quad (26)$$

Then, Eqn. (21) will become

$$E_{1m} = -2\pi m \theta \sum_{\alpha=1}^6 A^\alpha \quad (27)$$

Where, A^α are given by

$$A^1 = A_1 \int_0^{+\infty} r^{(2\delta+1)-1} \exp\left(-\sqrt{b}r^2 - \frac{a}{\sqrt{b}}r\right) dr \\ A^2 = A_2 \int_0^{+\infty} r^{2\delta-1} \exp\left(-\sqrt{b}r^2 - \frac{a}{\sqrt{b}}r\right) dr \\ A^3 = A_3 \int_0^{+\infty} r^{(2\delta-2)-1} \exp\left(-\sqrt{b}r^2 - \frac{a}{\sqrt{b}}r\right) dr \\ A^4 = A_4 \int_0^{+\infty} r^{(2\delta+2)-1} \exp\left(-\sqrt{b}r^2 - \frac{a}{\sqrt{b}}r\right) dr \\ A^5 = A_5 \int_0^{+\infty} r^{(2\delta-1)-1} \exp\left(-\sqrt{b}r^2 - \frac{a}{\sqrt{b}}r\right) dr \\ A^6 = A_6 \int_0^{+\infty} r^{(2\delta+3)-1} \exp\left(-\sqrt{b}r^2 - \frac{a}{\sqrt{b}}r\right) dr \quad (28)$$

Using the special integral (26), we obtain the following values

$$A^1 = A^1 = A_1 (2\sqrt{b})^{\frac{2\delta+1}{2}} \Gamma(2\delta+1) D_{-(2\delta+1)} \left(\frac{a}{\sqrt{2b^{\frac{3}{2}}}} \right) \\ A^2 = A_2 (2\sqrt{b})^{\frac{2\delta}{2}} \Gamma(2\delta) D_{-(2\delta)} \left(\frac{a}{\sqrt{2b^{\frac{3}{2}}}} \right) \\ A^3 = A_3 (2\sqrt{b})^{\frac{2\delta-2}{2}} \Gamma(2\delta-2) D_{-(2\delta-2)} \left(\frac{a}{\sqrt{2b^{\frac{3}{2}}}} \right) \\ A^4 = A_4 (2\sqrt{b})^{-(\delta+1)} \Gamma(2\delta+2) D_{-(\delta+1)} \left(\frac{a}{\sqrt{2b^{\frac{3}{2}}}} \right) \\ A^5 = A_5 (2\sqrt{b})^{\frac{(2\delta-1)}{2}} \Gamma(2\delta-1) D_{-(2\delta-1)} \left(\frac{a}{\sqrt{2b^{\frac{3}{2}}}} \right) \\ A^6 = A_6 (2\sqrt{b})^{\frac{(2\delta+3)}{2}} \Gamma(2\delta+3) D_{-(2\delta+3)} \left(\frac{a}{\sqrt{2b^{\frac{3}{2}}}} \right) \quad (29)$$

Which, allows us to obtain the non-commutative modification of the energy levels of the electron in the first excited states E_{1m}

$$E_{1m} = 2\pi m \theta \sum_{\alpha=1}^6 A^\alpha \quad (30)$$

Now, the non-commutative modification of the energy levels E_{pm} , in the first order of the corresponding p^{th} order excited states, will be determined by using Eqns. (10), (17) and (18)

$$E_{pm} \equiv 2\pi m \theta A \quad (31)$$

Where, A is given by

$$A = \int_0^{+\infty} \left[(a_0 + \dots a_p r^{2p}) \right]^2 r^{2\delta} \exp\left(-\frac{b}{2\sqrt{c}} r^2 - \frac{\sqrt{c}}{2} r^4\right) \exp\left(-\frac{a}{2r} - b + \frac{c}{2r^3}\right) dr \quad (32)$$

We summarize the obtained results for energy levels ($E_{NC0-m}, E_{NC1-m}, E_{NCp-m}$) corresponding to first order of θ , to the stationary state, the first excited states, and the p^{th} excited states, respectively as

$$E_{NC0-m} = E_0 + E_{0m} \\ E_{NC1-m} = E_1 + E_{1m} \\ E_{NCp-m} = E_p + E_{pm} \quad (33)$$

Then, we have the explicit obtained results as

$$\begin{aligned} E_{NC0-m} &= 2\sqrt{b}(1+m) - \theta F(a, b, c)m \\ E_{NC1-m} &= 2\sqrt{b}(2+m) + 2\pi\theta \left(\sum_{\alpha=1}^6 A^\alpha \right) m \\ E_{NCp-m} &= 2\sqrt{b}(1+m+p) + 2\pi m \theta A \end{aligned} \quad (34)$$

Now, in order to construct the NC Hamiltonian, this will be realized by two principal parts, the first is deformed NC Hamiltonian H_{hlc1} , which will be constructed based on Eqns. (8), (12) and (17), we can write

$$H_{hlc1} = \left[-\frac{\Delta}{2m_0} + \left(ar + br^2 + \frac{c}{r} \right) \right] + \left[\theta \left(-\frac{a}{2r} - b + \frac{c}{2r^3} \right) L_z \right] \quad (35)$$

For a better understanding of the physical content of the operator H_{hlc1} , we divide it to 2-operators, H_{hlc0} and H_{hlc-m} as follows

$$H_{hlc1} = H_{hlc0} + H_{hlc-m} \quad (36)$$

Where, the first operator is given by

$$H_{hlc0} = -\frac{\Delta}{2m_0} + ar + br^2 + \frac{c}{r} \quad (37)$$

This operator represents a particle with spin (1/2) interacted with the H.L.C. potential in the ordinary 2D space, as mentioned in Sec. 3. While the second operator H_{hlc-m} is given by

$$H_{hlc-m} = \theta \left(-\frac{a}{2r} - b + \frac{c}{2r^3} \right) L_z \quad (38)$$

If we choose the parameter θ and the magnetic field oriented in (Oz.) axes, we can obtain

$$\begin{cases} \theta = \alpha B \\ \vec{B} = B \vec{k} \\ \theta L_z = \alpha \vec{J} \vec{B} - \alpha \vec{S} \vec{B} \end{cases} \quad (39)$$

Where, $\vec{J}$ is the total momentum. Using Eqns. (38) and (39), we can write the operator H_{hlc-m} as

$$H_{hlc-m} = \alpha \left(-\frac{a}{2r} - b + \frac{c}{2r^3} \right) (\vec{J} \vec{B} - H_z) \quad (40)$$

Where, H_z represent the usual Zeeman effect in commutative space. The operator (Eqn. (40)) represents two interactions between a particle with spin (1/2) and an external magnetic field. The first one represents the ordinary Zeeman effect and the new interaction represents a coupling between the total momentum $\vec{J}$ and external magnetic field $\vec{B}$. Now, regarding to the relation (Eqn. (22)), which can be rewritten in the form

$$H_{hlc-p}(r) = \theta \left(-\frac{a}{r} - 2b + \frac{c}{r^3} \right) \vec{S} \vec{L} \quad (41)$$

Which allow writing the perturbative term $H_{hlc-p}(r)$ as follows

$$H_{hlc-p}(r) = \frac{\theta}{2} \left(-\frac{a}{r} - 2b + \frac{c}{r^3} \right) (\vec{J}^2 - \vec{L}^2 - \vec{S}^2) \quad (42)$$

Furthermore, the operator $(\vec{S} \vec{L})$ is traduced physically the coupling between spin and orbital momentum, then the modified energy levels $(E_{NC0-SO}, E_{NC1-SO}, E_{NCp-SO})$ produced will be

$$\begin{aligned} E_{NC0-SO} &= E_0 + \begin{cases} \theta L_{hlc1} F(a, b, c) \Rightarrow \text{spin up} \\ \theta L_{hlc2} F(a, b, c) \Rightarrow \text{spin down} \end{cases} \\ E_{NC1-SO} &= E_1 + \begin{cases} 2\pi \theta L_{hlc1} \sum_{\alpha=1}^6 A^\alpha \Rightarrow \text{spin up} \\ 2\pi L_{hlc2} \theta \sum_{\alpha=1}^6 A^\alpha \Rightarrow \text{spin down} \end{cases} \\ E_{NCp-SO} &= E_p + \begin{cases} 2\pi L_{hlc1} \theta A \Rightarrow \text{spin down} \\ 2\pi L_{hlc2} \theta A \Rightarrow \text{spin down} \end{cases} \end{aligned} \quad (43)$$

Where, L_{hlc1} and L_{hlc2} are given by

$$\begin{aligned} L_{hlc1} &= \left(l + \frac{1}{2} \right) \left(l + \frac{3}{2} \right) - l(l+1) - \frac{3}{4} \\ L_{hlc2} &= \left(l - \frac{1}{2} \right) \left(l + \frac{1}{2} \right) - l(l+1) - \frac{3}{4} \end{aligned} \quad (44)$$

Then, the second part of NC Hamiltonian and denoted by H_{hlc2} , which is given by the relation

$$H_{hlc2} = H_{hlc0} + H_{hlc-so} \quad (45)$$

Where, the operator of spin-orbital interaction H_{hlc-so} takes the form

$$H_{hlc-so} = \theta \left(-\frac{a}{r} - 2b + \frac{c}{r^3} \right) (\vec{J}^2 - \vec{L}^2 - \vec{S}^2) \quad (46)$$

On other hand, it is evident to consider infinitesimal parameter θ , the sum of two infinitesimal parameters θ_1 and θ_2 , then the complete NC Hamiltonian equals the sum of three fundamentals parts: the first one can be seen from Eqn. (38), the second can be seen from Eqn. (40), whereas the last part can be seen from Eqn. (46). Thus, the final expression of deformed NC Hamiltonian H_{hlc} takes the following

$$H_{hlc} = H_{hlc0} + H_{hlc-m} + H_{hlc-so} \quad (47)$$

It's worth mentioning that the above obtained NC Hamiltonian is satisfies the physical superposition principal. Thus, the obtained NC Hamiltonian was extended to describe, in addition to the usual interaction in 2D space, two new physical phenomena, the modified Zeeman effect and the spin orbital interaction, which are introduced in the new theory automatically. Regarding Eqn. (47), we can deduce the total NC quantum spectrum of Hydrogen atom with H.L.C. potential, $E_{NC}(j, l, s, m)$, which is evidently equal to the sum of three values of energies (E_0, E_1, E_2) , $(E_{NC0-SO}, E_{NC1-SO}, E_{NCp-SO})$ and $(E_{NC0-m}, E_{NC1-m}, E_{NCp-m})$ corresponding to $(H_{or}, H_{mag}$ and $H_{spin-orb})$, respectively, as follows

$$\begin{aligned} E_{NC0} &= E_0 + \begin{cases} F(a, b, c)(-\theta_1 m + \theta_2 L_{hlc1}) \Rightarrow \text{spin up} \\ F(a, b, c)(-\theta_1 m + \theta_2 L_{hlc2}) \Rightarrow \text{spin down} \end{cases} \\ E_{NC1} &= E_1 + \begin{cases} 2\pi \left(\sum_{\alpha=1}^6 A^\alpha \right) (m\theta_1 + \theta_2 L_{hlc1}) \Rightarrow \text{spin up} \\ 2\pi \left(\sum_{\alpha=1}^6 A^\alpha \right) (m\theta_1 + L_{hlc2} \theta_2) \Rightarrow \text{spin down} \end{cases} \\ E_{NCp} &= E_p + \begin{cases} 2\pi A(m\theta_1 + L_{hlc1} \theta_1) \Rightarrow \text{spin down} \\ 2\pi A(m\theta_1 + L_{hlc2} \theta_1) \Rightarrow \text{spin down} \end{cases} \end{aligned} \quad (48)$$

It is possible to construct the diagonal 2-elements $(H_{hlc})_{11}$ and $(H_{hlc})_{22}$ of the NC Hamiltonian matrices H_{hlc} , as it has been constructed in references [33,34] in the case of harmonic oscillator as

$$\begin{cases} (H_{hlc})_{11} = -\frac{\Delta}{2m_0} + \left(ar + br^2 + \frac{c}{r} \right) \\ \quad + \left(-\frac{a}{r} - 2b + \frac{c}{r^3} \right) \left(\alpha \vec{J} \vec{B} - \alpha H_z + \theta L_{hlc1} \right) \\ (H_{hlc})_{22} = -\frac{\Delta}{2m_0} + \left(ar + br^2 + \frac{c}{r} \right) \\ \quad + \left(-\frac{a}{r} - 2b + \frac{c}{r^3} \right) \left(\alpha \vec{J} \vec{B} - \alpha H_z + \theta L_{hlc2} \right) \end{cases} \quad (49)$$

Now, it's possible to give the number of energy levels for a fixed quantum orbital number l , we have $(2l+1)$ possible values of $(m=l, l-1, \dots, -l)$ if we consider the spin-orbital effect, the eigenvalues of the operator $\vec{J}$ equal $j = l \pm \frac{1}{2}$, which allow us to consider $2(2l+1)$ values of energy for a given orbital number l . Finally, we examine our obtained NC physical results and by the asymptotic limit of the parameter $\theta \rightarrow 0$, we can obtain all the ordinary 2D space results.

5. Conclusion

In this paper, we solved the Schrödinger equation for H.L.C. potential by applying Boopp's shift method to first order of the parameter θ in non-commutative two-dimensional spaces, the modified energy levels at the fundamental state and the first excited states are established. We have seen that the non-commutativity produced three sub-Hamiltonian operators, the first one naturally to describe the usual H_{hlc10} , the second sub-Hamiltonian H_{hlc-m} describes the new modified Zeeman effect, while the last sub-Hamiltonian $H_{spin-orb}$ corresponds to spin-orbital interaction. Furthermore, the new energy spectrum is changed radically; it depends now on four quantum numbers $\left(j = l \pm \frac{1}{2}, l, s \right)$ and m . For a fixed number of quantum orbital number l , we found that the number of energy states equals $2(2l+1)$.

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References

- [1] D. Shi-Ha, Int. J. Theor. Phys. **40**, 2, 559 (2001).
- [2] R. L. Liboff, *Introductory Quantum Mechanics*, fourth edition (Addison Wesley, San Francisco, CA, 2003).
- [3] Shi-Hai Dong and M. Lozada-Cassou, American J. Applied Sciences **2**, 1, 376 (2005).
- [4] E. Elbaz, *Quantum*, first edition (Springer-Verlag, Berlin, Heidelberg, New York, 1998).
- [5] T. Chang and L. Fong Li, *Gauge Theory of Elementary Particle Physics*, first edition (Clarendon Press, Oxford, USA, 1988).
- [6] G. Vanden Berghe, H. De Meyer and M. Van Daele, Int. J. Theor. Phys. **34**, 9, 1931 (1995).
- [7] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics Relativistic Theory*, third edition (Cambridge University Press, London, 1997).
- [8] Shi-Hai Dong and Guo-Hua San, Found. Phys. Lett. **16**, 4, 357 (2003).
- [9] Shi. Hai Dong. Int. J. Theor. Phys. **39**, 4, 1119 (2000).
- [10] Sameer M. Ikhdair, Ramazan Sever. Cent. Eur. J. Phys. **6**, 3, 685 (2008).
- [11] G. Q. Hassoun, Am. J. Phys. **49**, 2, 143 (1981).
- [12] D. Shi-Hai, Found. Phys. Lett. **15**, 4, 385 (2002).
- [13] G. R. Khan, Eur. Phys. J. D **53**, 123 (2009).
- [14] D. Shi-Hai, Int. J. Theor. Phys. **39**, 4, 1119 (2000).
- [15] D. Shi-Hai, Found. Phys. Lett. **15**, 4, 385 (2002).
- [16] L. Buragohain and S. A. S. Ahmed, Lat. Am. J. Phys. Educ. **4**, 1, 79 (2010).
- [17] Nabaratna Bhagawati, Cent. Eur. J. Phys. **12**, 4, 256 (2014).
- [18] Yuan Yi, Li Kang, Wang Jain-Hua and Chen Chi-Yi, Chinese Physics C **34**, 5, 543 (2010).
- [19] K. A. Gepreel and A. A. Al-Thobaiti, Indian J. Phys. **88**, 3 (2014).
- [20] Zaim Slimane, Int. J. Theor. Phys. **53**, 2014 (2014).
- [21] B. Mirza and M. Mohadesi, Commun. Theor. Phys. **42**, 664 (2004).
- [22] H. S. Snyder, Phys. Rev. **71**, 38 (1947).
- [23] Y. I. Manin, Math. Phys. **123**, 163. (1989).
- [24] N. Seiberg and E. Witten, JHEP **09**, 032 (1999).
- [25] J. Madore, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. **C16**, 161 (2000).
- [26] J. H. Wang, K. Li and S. Dulat, Chinese Phys. C. **32**, 10, 803 (2008).
- [27] B. Mirza and M. Zarei, Eur. Phys. J. C. **32**, 583 (2004).
- [28] S. Belluchi, Phys. Lett. B. **552**, 345 (2001).
- [29] C. Acatrinei, JHEP **0109**, 007 (2001).
- [30] M. Chaichian, M. M. Sheikh-Jabbari and A. Tureanu, Phys. Rev. Lett. **86**, 2716 (2001).
- [31] A. Connes, M. R. Douglas and A. Schwarz, JHEP **9802**, 003 (1998); hep-th/9808042.
- [32] N. G. Deshpande, Parama-J. Phys. **60**, 2, 189 (2003).
- [33] Abdelmadjid Maireche, The African Review of Physics **9**, 185 (2014).
- [34] Abdelmadjid Maireche, Life Science J. **11(6)**, 353 (2014).
- [35] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, seventh edition (Elsevier, 2007).

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