

Micro Stiff Fluid Cosmological Constant Model in Bianchi Type-1 Self Creation Cosmology

R. C. Sahu^{1,*}, S. P. Misra² and Ritarani Swain³

¹*Department of Mathematics, K.S.U.B. College, Bhanjanagar-761126, Odisha, India*

²*Department of Mathematics, Sri Jagannath Mahavidyalaya, Rambha, Ganjam, Odisha, India*

³*Department of Physics, U.P.S.C. College, Sheragada,-761106, Ganjam, Odisha, India*

In the present paper, Bianchi type-1 micro cosmological model for perfect fluid distribution in the presence of cosmological constant is investigated in Barber's second self-creation theory of gravitation. The role of cosmological constant- Λ for three different cases, i.e., $\Lambda = 0$, $\Lambda > 0$ and $\Lambda < 0$ is studied in order to determine solution of the field equations. In first case i.e., when $\Lambda = 0$, it is seen that the result reduces to that of found by Mohanty et al. (2003). In second case, i.e., when $\Lambda > 0$, it is observed that the real physical model of the universe does not exist. However, in third case i.e., when $\Lambda < 0$, the perfect fluid characterized by the equation of state $p = \rho$ degenerates homogeneous, inflationary and isotropic model in second self-creation theory.

1. Introduction

It is known that theory of general relativity proposed by Einstein [2] is a coordinate invariant theory and it serves as a basis for constructing mathematical models of the universe. In recent years researchers have taken some interesting attempts to generalise this theory by incorporating Mach's principle and other desired features which are lacking in original theory. Thus several modifications of Einstein's theory are proposed by authors from time to time. Second self-creation theory proposed by Barber [3] is a modification of general relativity to a variable G-theory and predicts local effects that are within the observational limits. In this theory the scalar field does not gravitate directly, but simply divides the matter tensor, acting as a reciprocal to gravitational constant. In the limit $\lambda \rightarrow 0$, this theory tends to Einstein's theory in every respect. Some of the authors who have studied second self-creation theory in various angles taking different space times in presence of different gravitating fields are Pimentel [4], Soleng [5,6], Venkateswarlu and Reddy [7], Shanti and Rao [8], Carvalho [9], Shri Ram and Singh [10], Mohanty et al. [11-13], Panigrahi and Sahu [14-16], Sahu and Panigrahi [17], Sahu and Mohanty [18], Sahu and Mahapatra [19,20] and recently Sahu et al. [21].

The cosmological constant Λ has been introduced in 1917 by Einstein to modify his own equations of general relativity. Now this Λ term remains a focal point of interest in the context of quantum field theories, quantum gravity, super gravity theories, Kaluza-Klein theories and in the inflationary universe scenario. A number of observations suggest that the universe possess a non-zero cosmological constant [22]. The cosmological term which is a measure of the energy of empty space, provides a repulsive force opposing the gravitational pull between the galaxies. If it exists, then the energy it represents counts as mass because Einstein has shown that mass and energy are equivalent. Moreover, if the cosmological constant is large value then the energy involved due to the matter in the universe, can sum up to the number that inflation predicts. In this inflationary era it is quite important to study about the cosmological constant. But recent research suggests that the cosmological term corresponds to a very small value of the order 10^{-58} cm^{-2} [23].

The scalar meson fields represent matter fields with spin less quanta. There are two types of scalar fields, viz. zero rest mass scalar fields and massive scalar fields. The zero rest mass scalar fields describe long range interactions, whereas massive scalar fields describe short range interactions. The study of scalar meson fields in general relativity has a focal point for the researchers due to its physical importance in particle physics. The massless scalar field in relativistic mechanics

*rcsahu2@rediffmail.com

yields some significant results as regards to the singularities. The scalar meson field being a field of a single variable 'v' (say) is the special case of general field and the expression given by

$$T_{ij} = v_i v_j - \frac{1}{2} g_{ij} (v_k v^k - m^2 v^2)$$

is the energy-momentum tensor of Yukawa [24] fields (spin zero meson particle) for the metric of (+2) signature in flat space time.

The Klein-Gorden equation takes the form

$$g^{ij} v_{;ij} + m^2 v = 0$$

Where, 'v' is the real scalar field and m is the rest-mass of scalar meson field. Here (;) semicolon followed by an index denotes covariant differentiation. When $m = 0$, the scalar field 'v' is known as massless scalar field or micro matter field.

So far none of the authors has studied Bianchi type-1 space time in the context of second self-creation theory when the gravitational field in presence of cosmological constant is a mixture of massless scalar field and perfect fluid. So in the present paper, we are interested to study the role of Λ (cosmological constant) in deriving mesonic perfect fluid solutions for the spatially homogeneous and anisotropic Bianchi type-1 space time in Barber's second self-creation theory. The advantage of introducing Λ here is to reveal the cosmological characters of the nature. In section 2, we have obtained the field equations and solved the field equations corresponding to three different cases taken in section 3. In first case i.e. when $\Lambda = 0$ the result reduces to that of found by Mohanty et al. [1]. In second case, i.e., when $\Lambda > 0$ the real physical model of the universe does not exist. However, in third case i.e., when $\Lambda < 0$ the perfect fluid characterized by the equation of state $p = \rho$ degenerates homogeneous isotropic model. In section 4, some physical and geometrical properties of the solutions are studied. The work presented in this paper is treated as an extension work of Mohanty et al. [1], Sahu and Panigrahi [17] and Sahu and Mahapatra [19,20].

2. Field Equations

We have considered Bianchi type-I metric in the form

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 dy^2 + C^2 dz^2 \quad (1)$$

Where, A, B and C are functions of cosmic time 't' which ensures that the space-time is spatially homogeneous.

The Einstein-Barber's field equations for gravitating mesonic perfect fluid with cosmological term Λg_{ij} may be written as

$$G_{ij} \equiv R_{ij} - \frac{1}{2} g_{ij} R + \Lambda g_{ij} = -8\pi\phi^{-1} (T_{ij}^p + T_{ij}^v) \quad (2)$$

and

$$\phi = \frac{8\pi}{3} \lambda T \quad (3)$$

Where, $\square \phi = \phi_{;k}^{;k}$ is invariant D' Alembertian, T is the trace of the energy-momentum tensor $(T_{ij}^p + T_{ij}^v)$, ϕ is the Barber's scalar and λ is the coupling constant to be evaluated from the experiment. The measurements of the deflection of light restricts the value of coupling to $|\lambda| \leq 10^{-1}$. In the limit $\lambda \rightarrow 0$ this theory approaches the Einstein's general relativity theory in every respect.

The energy momentum tensor T_{ij}^p for perfect fluid distribution is given by

$$T_{ij}^p = (\rho + p)u_i u_j - p g_{ij} \quad (4)$$

together with

$$g_{ij} u^i u^j = -1 \quad (5)$$

Here ρ , p and u_i are respectively the mass energy density, isotropic pressure and four-velocity vector of the perfect fluid.

The energy momentum tensor T_{ij}^v for a micro matter field representing massless scalar field distribution is taken as

$$T_{ij}^v = v_i v_j - \frac{1}{2} g_{ij} v_k v^k \quad (6)$$

together with

$$g^{ij} v_{;ij} = 0 \quad (7)$$

Here the scalar field v is a function of cosmic time.

By adopting co-moving coordinates the field Eqns. (2) and (3) for the metric (1) and the Klein-Gorden Eqn. (7) for the metric (1), obtained as

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} + \Lambda = -8\pi \phi^{-1} \left(p + \frac{v_4^2}{2} \right) \quad (8)$$

$$\frac{C_{44}}{C} + \frac{A_{44}}{A} + \frac{C_4 A_4}{CA} + \Lambda = -8\pi \phi^{-1} \left(p + \frac{v_4^2}{2} \right) \quad (9)$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} + \Lambda = -8\pi \phi^{-1} \left(p + \frac{v_4^2}{2} \right) \quad (10)$$

$$\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{C_4 A_4}{CA} + \Lambda = 8\pi \phi^{-1} \left(p + \frac{v_4^2}{2} \right) \quad (11)$$

$$\phi_{44} + \phi_4 \left[\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \right] = -\frac{8\pi}{3} \lambda (3p - \rho + v_4^2) \quad (12)$$

and

$$v_{44} + [\log(ABC)]_4 v_4 = 0 \quad (13)$$

Where, the subscript '4' denotes the ordinary differentiation with respect to time.

3. Cosmological Model

The system of field Eqns. (8) to (10) is an underdetermined system having six equations in eight unknowns viz., A, B, C, p, ρ, φ, Λ and v. So, in order to make the system consistent, we have to consider two additional conditions.

Let us consider the barotropic equation of state

$$p = \gamma \rho, \quad 0 \leq \gamma \leq 1 \quad (14)$$

as first additional condition.

3.1. Mesonic stiff fluid model

Taking γ = 1, Eqn. (14) reduces to

$$p = \rho \quad (15)$$

which represents stiff fluid [25]. In this case adding Eqns. (8), (9) and (10) with three times of Eqn. (11), we obtained

$$\frac{(ABC)_{44}}{ABC} + 3\Lambda = 12\pi \phi^{-1} (\rho - p) \quad (16)$$

On integration Eqn.(13) yields

$$v_4 = \frac{a_1}{ABC} \quad (17)$$

Where, $a_1 \neq 0$ is the constant of integration.

Now use of Eqn. (15) in Eqns. (16) and (12), we get

$$(ABC)_{44} + 3\Lambda ABC = 0 \quad (18)$$

and

$$\phi_{44} + \phi_4 \frac{(ABC)_4}{ABC} = \frac{-16\pi\lambda}{3} \left(\rho + \frac{v_4^2}{2} \right) \quad (19)$$

3.2. Solutions

In order to get exact and explicit solutions, we consider the following three cases, i.e., (i) Λ = 0 (ii) Λ > 0 (iii) Λ < 0 as second additional condition.

Case - i: For Λ = 0, the result reduces to that of obtained by Mohanty et al. [1].

Case - ii: For Λ > 0, Eqn. (18) after integration yields

$$ABC = \alpha_1 \cos \sqrt{3\Lambda} t + \alpha_2 \sin \sqrt{3\Lambda} t \quad (20)$$

Where, α_1 and α_2 are constants of integration. From Eqn. (20), we can write the explicit form for A, B and C as

$$\left. \begin{aligned} A &= \left(\alpha_1 \cos \sqrt{3\Lambda} t + \alpha_2 \sin \sqrt{3\Lambda} t \right)^{n_1}, \\ B &= \left(\alpha_1 \cos \sqrt{3\Lambda} t + \alpha_2 \sin \sqrt{3\Lambda} t \right)^{n_2} \\ \text{and} \\ C &= \left(\alpha_1 \cos \sqrt{3\Lambda} t + \alpha_2 \sin \sqrt{3\Lambda} t \right)^{n_3} \end{aligned} \right\} \quad (21)$$

Where, n_i , $i = 1, 2, 3$ are real constants and satisfies the relation

$$\sum_{i=1}^3 n_i = 1 \quad (22)$$

Here the over determinacy for determining three unknowns A, B and C from four field Eqns. (8)-(11) can be settled by actual substitution of the values of A, B and C from Eqn. (21) in Eqn. (11). Thus we obtain

$$\sum_{\substack{i,j=1 \\ i \neq j}}^3 n_i n_j = \frac{[8\pi \phi^{-1} \rho - \Lambda]}{3\Lambda} \cdot \left[\frac{\alpha_1 \cos \sqrt{3\Lambda} t + \alpha_2 \sin \sqrt{3\Lambda} t}{\alpha_2 \cos \sqrt{3\Lambda} t - \alpha_1 \sin \sqrt{3\Lambda} t} \right]^2 \quad (23)$$

As $n_i, i = 1, 2, 3$ are real constants, so also $\sum_{\substack{i,j=1 \\ i \neq j}}^3 n_i n_j$

is a real constant. But this relation cannot be hold good in Eqn. (23) as its L.H.S part is constant but R.H.S part is a function of 't'.

Thus for $\Lambda > 0$ and $p = \rho$, it is not possible to determine real physical model of the universe in Barber's second self-creation theory.

Case - iii: Suppose $\Lambda < 0$ i.e., $\Lambda = \frac{-\omega^2}{3}$. In this case Eqn. (18) reduces to

$$(ABC)_{44} - \omega^2 ABC = 0 \quad (24)$$

On integration, Eqn. (24) yields

$$ABC = \beta_1 e^{\omega t} + \beta_2 e^{-\omega t} \quad (25)$$

Where, β_1 and β_2 are arbitrary constants of integration. From Eqn. (25), we can write the explicit form for A, B and C as

$$\left. \begin{aligned} A &= (\beta_1 e^{\omega t} + \beta_2 e^{-\omega t})^{r_1}, \\ B &= (\beta_1 e^{\omega t} + \beta_2 e^{-\omega t})^{r_2} \\ \text{and} \\ C &= (\beta_1 e^{\omega t} + \beta_2 e^{-\omega t})^{r_3} \end{aligned} \right\} \quad (26)$$

Where, $r_i, i = 1, 2, 3$ are real constants satisfying the condition

$$\sum_{i=1}^3 r_i = 1 \quad (27)$$

Here also the over determinacy for finding three unknowns A, B and C from four Eqns. (8) to (11) can be settled by putting the values of A, B and C from Eqn. (26) in Eqn. (11). Thus we get

$$\sum_{\substack{i,j=1 \\ i \neq j}}^3 r_i r_j = \frac{1}{3} \cdot \left[\frac{\beta_1 e^{\omega t} + \beta_2 e^{-\omega t}}{\beta_1 e^{\omega t} - \beta_2 e^{-\omega t}} \right]^2 \cdot \left[\frac{8\pi\phi^{-1} \left(\rho + \frac{v_4^2}{2} \right) - \Lambda}{-\Lambda} \right] \quad (28)$$

Since $r_i, i = 1, 2, 3$ are real constants, so $\sum_{\substack{i,j=1 \\ i \neq j}}^3 r_i r_j$

is also a real constant. But this relation holds good in Eqn. (28) only when

$$\beta_2 = 0 \quad \text{and} \quad \frac{8\pi\phi^{-1} \left(\rho + \frac{v_4^2}{2} \right) - \Lambda}{-\Lambda} = 1 \quad (29)$$

Thus from Eqns. (28) and (29), we obtain

$$\sum_{\substack{i,j=1 \\ i \neq j}}^3 r_i r_j = \frac{1}{3} \quad (30)$$

and

$$8\pi\phi^{-1} \left(\rho + \frac{v_4^2}{2} \right) = 0 \quad (31)$$

From Eqn. (31), we get

$$\rho = -\frac{v_4^2}{2} \quad (32)$$

Thus from (32) and (15), we have

$$\rho = p = -\frac{v_4^2}{2} \quad (33)$$

Use of (33) in Eqn. (19), we find

$$\phi_{44} + \phi_4 \frac{(ABC)_4}{ABC} = 0 \quad (34)$$

On integration (34) yields

$$\phi_4 = \frac{a_2}{ABC} \quad (35)$$

Where, $a_2 \neq 0$ is a constant of integration.

Now putting the value of $\beta_2 = 0$ from Eqn. (29) in Eqn. (25), we obtain

$$ABC = \beta_1 e^{\omega t} \quad (36)$$

Solving (27) and (30), one can get

$$r_1 = r_2 = r_3 = 1/3 \quad (37)$$

In view of (30) putting the value of $\beta_2 = 0$ from (29), the explicit expressions of A, B and C in (26) can be expressed as

$$A = B = C = (\beta_1)^{1/3} \cdot e^{\frac{\omega t}{3}} \quad (38)$$

Using Eqn. (36) in Eqns. (35) and (17), we obtain

$$\phi = \frac{-a_2}{\beta_1 \omega} \cdot \frac{1}{e^{\omega t}} + a_3 \quad (39)$$

and

$$v = \frac{-a_1}{\beta_1 \omega} \cdot \frac{1}{e^{\omega t}} + a_4 \quad (40)$$

Where, a_3 and a_4 are constants of integration.

With the help of Eqn. (40), Eqn. (33) yields

$$\rho = p = -\frac{a_1^2}{2\beta_1^2} \cdot \frac{1}{e^{2\omega t}} \quad (41)$$

Thus corresponding to our solution (38) the Bianchi type-1 metric (1) can be expressed as

$$ds^2 = -dt^2 + (\beta_1 e^{\omega t})^{2/3} (dx^2 + dy^2 + dz^2) \quad (42)$$

The model obtained as above is the spatially-homogeneous isotropic Bianchi type-1 stiff fluid micro model in Barber's second theory. This model does not lead to well-known Kasner [26] model for $\Lambda < 0$.

3.3. Vacuum model

When $\gamma = 0$ and $\rho = 0$, Eqn. (14) reduces to

$$p = \rho = 0 \quad (43)$$

In this case the result reduces to that of already studied by Sahu and Mahapatra [19,20].

4. Some Physical and Geometrical Features of the Model

In this section we intended to study the following properties of the model (42).

- i. The spatial volume in the model is given by $V = (-g)^{1/2} = ABC = \beta_1 e^{\omega t}$.

Thus $V \rightarrow \beta_1$ as $t \rightarrow 0$ and $V \rightarrow \infty$ as $t \rightarrow \infty$. Thus the universe starts expanding from a constant volume and becomes infinite large volume as $t \rightarrow \infty$. Hence we concluded that the universe may blows up at infinite future.

- ii. The scalar expansion θ and the anisotropy $|\sigma|$ are defined by [27] as

$$\theta = u^i_{;i} = \frac{V_4}{V}$$

Where, V the volume element and

$$\sigma^2 = \frac{1}{12} \left[\left(\frac{g_{11,4}}{g_{11}} - \frac{g_{22,4}}{g_{22}} \right)^2 + \left(\frac{g_{22,4}}{g_{22}} - \frac{g_{33,4}}{g_{33}} \right)^2 + \left(\frac{g_{33,4}}{g_{33}} - \frac{g_{11,4}}{g_{11}} \right)^2 \right]$$

Now the scalar expansion θ and anisotropy $|\sigma|$ in the model we found

$$\theta = \omega > 0 \text{ and } \sigma^2 = 0. \text{ Thus } \lim_{t \rightarrow \infty} \frac{\sigma}{\theta} = 0.$$

The above result shows that the universe is expanding in nature with constant expansion. Also the universe is non-shearing and isotropic throughout the evolution. Hence the spatially homogeneous anisotropic cosmological model in Barber's second theory reduces to homogeneous isotropic model. If

- iii. The Barbers scalar ϕ in the model is found to be

$$\phi = \frac{-a_2}{\beta_1 \omega} \cdot \frac{1}{e^{\omega t}} + a_3$$

Thus $\phi \rightarrow$ a constant as $t \rightarrow 0$ and $\phi \rightarrow a_3$ as $t \rightarrow \infty$. So Barber's scalar has no singularity in this model. If $a_2 = 0$ then ϕ will be constant. Thus Barber's theory leads to Einstein theory.

- iv. The Kretschmann curvature invariant defined by $L = R_{hijk} R^{hijk}$, where R_{hijk} is the Riemann curvature tensor. Here L is found to be

$$L = \frac{2}{3} (\Lambda^2) > 0$$

As L is a + ve constant, the result confirms that the model has no geometrical singularity.

- v. The massless scalar field v in this model is found to be

$$v = \frac{-a_1}{\beta_1 \omega} \cdot \frac{1}{e^{\omega t}} + a_4$$

Thus $v \rightarrow$ a constant as $t \rightarrow 0$ and $v \rightarrow a_4$ as $t \rightarrow \infty$. If $a_1 = 0$, then $v = a_4$.

In this case the massless scalar field v does not exist.

- vi. The energy density and proper pressure in the model are given by

$$\rho = p = -\frac{a_1^2}{2\beta_1^2} \cdot \frac{1}{e^{2\omega t}}$$

Here $\rho (= p) \rightarrow$ a -ve constant as $t \rightarrow 0$ and $\rho (= p) \rightarrow 0$ as $t \rightarrow \infty$.

Thus it is evident from the result that at initial time the result leads to unphysical situation but at infinite future the model shows singularity. If $a_1 = 0$, then $\rho = p$ will be constant. In this case the space time reduces to flat space time.

- vii. The Hubble's parameter H in the model is found as $H = \frac{\omega}{3}$. Thus H is not a function of time and hence we concluded that the model is of steady-state.
- viii. The scale factor S^3 in the model can be determined as $S^3 = ABC = \beta_1 e^{\omega t}$. So, S increases as time increase.
- ix. The deceleration parameter 'q' in the model is found to be $q = -\frac{VV_{44}}{V_4^2} = -1$. As $q = -1$, so the model of the universe corresponds to an inflationary model.

5. Conclusion

In this paper, we studied the role of Λ (cosmological constant) in deriving mesonic perfect fluid solutions for the spatially homogeneous anisotropic Bianchi type-1 space time in Barber's second self-creation theory corresponding to three different cases. In first case i.e., when $\Lambda = 0$ the result which is to be come out are same as that of found and studied by Mohanty et al. [1]. In second case, i.e., when $\Lambda > 0$, it has shown that the real physical model of the universe does not exist. However, in third case i.e., when $\Lambda < 0$, the perfect fluid characterized by the equation of state $p = \rho$ degenerates spatially homogeneous anisotropic Bianchi type-1 model in Barber's second theory which reduces to spatially homogeneous isotropic model in Einstein theory. It is also observed that the model found in Barber's modified theory is inflationary, uniformly expanding, non-shearing and has no geometrical singularity. Thus basic aim of Barber was fully fulfilled in the present work.

The work presented in this paper is treated as an extension work of Mohanty et al. [1], Sahu and Panigrahi [17] and Sahu and Mahapatra [19,20].

References

- [1] G. Mohanty, R. C. Sahu and U. K. Panigrahi, *Astrophys. Space Sci.* **284**, 1055 (2003).
- [2] A. Einstein, *Ann. Physik* **49**, 769 (1916).
- [3] G. A. Barber, *Gen. Rel. Grav.* **14**, 117 (1982).
- [4] L. O. Pimentel, *Astrophys. Space Sci.* **116**, 395 (1985).
- [5] H. H. Soleng, *Astrophys. Space Sci.* **137**, 373 (1987).
- [6] H. H. Soleng, *Astrophys. Space Sci.* **139**, 13 (1987).
- [7] R. Venkateswarlu and D. R. K. Reddy, *Astrophys. Space Sci.* **168**, 193 (1990).
- [8] K. Shanti and V. U. M. Rao, *Astrophys. Space Sci.* **179**, 147 (1991).
- [9] J. C. Carvalho, *Int. J. Theor. Phys.* **35**, 2019 (1996).
- [10] Shri Ram and C. P. Singh, *Astrophys. Space Sci.* **257**, 123 (1998).
- [11] G. Mohanty, B. Mishra and Reeta Das, *Bull. Inst. Math. Academia Sinica (ROC)*, **28**, 43 (2000).
- [12] G. Mohanty, U. K. Panigrahi and R. C. Sahu, *Astrophys. Space Sci.* **281**, 633 (2002).
- [13] G. Mohanty, R. C. Sahu and U. K. Panigrahi, *Orissa Math. Soc.* **23-24**, 293 (2004).
- [14] U. K. Panigrahi and R. C. Sahu, *National Aca. Sc. Lett.* **25**, 11 (2002).
- [15] U. K. Panigrahi and R. C. Sahu, *Theor. Appl. Mech.* **30**, 163 (2003).
- [16] U. K. Panigrahi and R. C. Sahu, *Czech. Jr. Phys.* **54**, 543 (2004).
- [17] R. C. Sahu and L. K. Mahapatra, *J.I.A.P.S.* **13**, 39 (2009).
- [18] R. C. Sahu and L. K. Mahapatra, *J.I.A.M* **31**, 307 (2009).
- [19] R. C. Sahu and G. Mohanty, *Astrophys. Space Sci.* **306**, 179 (2006).
- [20] R. C. Sahu and U. K. Panigrahi, *Astrophys. Space Sci.* **288**, 601 (2003).
- [21] R. C. Sahu, L. K. Mahapatra and G. Mohanty, *Romanian Reports in Phys.* **62**, 249 (2010).
- [22] L. M. Krauss and M. S. Turner, *Gen. Rel. Grav.* **27**, 1137 (1995).
- [23] V. B. Jhori and R. Chandra, *J. Math. Phys. Sci.* **17**, 73 (1983).
- [24] H. Yukawa, *Modern Physics*, Ed. J. C. Slater, Asian Student Edition (McGraw-Hill Book Inc., 1935) p.301.

- [25] Ya. B. Zel'dovich, Soviet Phys. JETP, **14**, 1143 (1962).
- [26] E. Kasner, Am. J. Math. **43**, 217 (1921).
- [27] A. K. Roychaudhuri, Phys. Rev. **98**, 1123 (1955).

Received: 26 October, 2014

Accepted: 4 February, 2015