

Bianchi Type–III Cosmological Models with Bulk Viscosity in $f(R)$ Theory

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The paper is devoted to a study of the Bianchi type-III space-time filled with bulk viscous in the framework of $f(R)$ gravity. We have presented three classes of cosmological models. We have obtained two classes of physically viable cosmological models $n \neq 1$ and $n = 1$. For the first model i.e., $n \neq 1$, the model evolves with a finite volume at $t = 0$. For the case of $n = 1$, the model starts evolving from a big-bang singularity at $t = 0$. The third model initially accelerates for a certain period of time and decelerates thereafter. The physical and kinematical features of the models are studied and discussed. The function $f(R)$ of the Ricci scalar is also evaluated for these models.

1. Introduction

Recent observations [1-4] including Type Ia Supernovae [5] have suggested the current cosmic expansion is accelerating. For the universe to be strictly homogeneous and isotropic, there are two major approaches: To introduce dark energy within general relativity (for reviews, see [6]) and to modify gravity for large distances (for recent reviews, see [7]). Furthermore, it was realized that modified gravity can describe dark energy [8,9] and also unify dark energy era with early-time cosmic acceleration [10]. In order to explain the nature of dark energy and the accelerated expansion, a variety of theoretical models have been proposed in the literature, such as quintessence [11], phantom energy [12], k-essence [13], tachyon [14], f-essence [15], Chaplygin gas [16], g-essence, etc. Among these different models of dark energy, the modified gravity models are quite interesting as they incorporate some notions from quantum and classical gravity theories. There are several modified gravity theories like $f(R)$ gravity, $f(G)$ gravity, $f(T)$ gravity and so on (see e.g., Refs.[17]-[19]).

Fourth-order theories of gravity have attracted huge attention recently, mainly because of their potential candidacy for dark energy-inspired cosmic acceleration and early universe inflation. These are a class of higher-order gravity models that attempt to address the shortcomings of General Relativity in the infrared (IR) and ultraviolet (UV)

ranges, that is, for very low and very high energy scales [20-22]. They are generally obtained by including higher order curvature invariants in the Einstein-Hilbert action, or by making the action non-linear in the Ricci curvature R and/or contain terms involving combinations of derivatives of R , in which case the models are known as $f(R)$ theories of gravity. First proposed by Buchdal [23], $f(R)$ theories gained more popularity after further developments by Starobinsky [24] and later following the realization of the discrepancy between theory and observation [25-34]. Since $f(R)$ gravity theories do generally have enough freedom to produce any kind of cosmic background evolution history or to explain some aspects of cosmological phenomena or another by a suitable choice of the defining action functional, a consistency analysis of the field equations describing those phenomena seems in order.

Lobo and Oliveira [35] constructed wormhole geometries in the context of $f(R)$ theories of gravity. Cognola et al. [36] investigated $f(R)$ gravity at one-loop level in de-Sitter universe. It was found that one-loop effective action can be useful for the study of constant curvature black hole nucleation rate. Multamäki and Vilja [37] investigated spherically symmetric vacuum solutions in $f(R)$ theory. The same authors [38] also studied the perfect fluid solutions and showed that pressure and density did not uniquely determine $f(R)$. Capozziello et al. [39] explored spherically symmetric solutions of $f(R)$ theories of gravity via the Noether symmetry approach. Hollenstein and Lobo [40] analyzed exact solutions of static spherically symmetric space-times in $f(R)$ gravity

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coupled to non-linear electrodynamics. M. Shamir [41] explored the vacuum solutions of Bianchi types I, III and Kantowski-Sachs space-times in metric $f(R)$ gravity. For some special cases, it is found that $f(R)$ models include some root powers of the Ricci scalar.

S. Asgari and R. Saffari [42] study consistent solutions of spherically symmetric space in metric $f(R)$ gravity theory. Shojai and Shojai [43] calculated exact spherically symmetric interior solutions in metric version of $f(R)$ gravitational theory. Azadi et al. [44] investigated cylindrically symmetric vacuum solutions in this theory. Plane symmetric solutions are studied by Sharif and Shamir [45]. The same authors [46-47] investigated the solutions for Bianchi types I and V models for both vacuum and non-vacuum case. Amare Abebe [48] investigated a class of perfect-fluid “anti-Newtonian” cosmological models in the context of $f(R)$ gravity. M. Khurshudyan et al. [49] suggested three different models of dark energy with variable gravitational constant G and Λ in the context of higher order $f(R)$ gravity. Sebastian and Kuriakose [50] have studied the scattering of scalar field around an extended black hole in $f(R)$ gravity using WKB method. Paul and collaborators [51] studied FRW cosmologies in $f(R)$ gravity.

In the early stages of evolution of the universe, the dissipative processes play a significant role for the high degree of isotropy we observe today. The inclusion of dissipative term in the energy-momentum of cosmic fluid seems to be the best generalization of matter term of the gravitational field equations. Eckart [52] developed the first relativistic theory of non-equilibrium thermodynamics to the effect of bulk viscosity. Padmanabhan and Chitre [53] investigated that the presence of bulk viscosity leads to inflationary like solutions in general relativity. Romano and Pavon [54], Saha [55-57] investigated the evolution of Bianchi type I universe for viscous fluid distribution. The effect of bulk viscosity on cosmological evolution has also been studied by Singh et al. [58], Sahni and Starobinski [59], Peebles [60], Bali and Sharma [61], Bali and Pradhan [62], Bali and Kumawat [63], Bali [64], Fabris et al. [65], Datta Chaudhary, and Sil [66].

After noting such recent developments in $f(R)$ gravity, the authors aimed to study different cosmological models in $f(R)$ gravity with bulk viscous. The main objective of this work is to find solutions of the field equations of Bianchi type-III model with Bulk Viscous in $f(R)$ gravity.

2. $f(R)$ Gravity Formalism

The $f(R)$ theory of gravity is the generalization of General Relativity. The action for this theory is given by

$$S = \frac{1}{2k^2} \int d^4 \sqrt{-g} f(R) + \int d^4 x L_m(g_{\mu\nu}, \psi_m) \quad (1)$$

Here $f(R)$ is a general function of the Ricci scalar, $k^2 = 8\pi G = 1$, g is the determinant of the metric $g_{\mu\nu}$ and L_m is the metric Lagrangian that depends on $g_{\mu\nu}$ and the matter field ψ_m .

It is noted that this action is obtained just by replacing R by $f(R)$ in the standard Einstein-Hilbert action. There are two formalisms which are applied to derive the field equations in $f(R)$ gravity. One is the standard metric formalism while another is Palatini formalism.

The corresponding field equations are found by varying the action with respect to the metric $g_{\mu\nu}$

$$F(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \square F(R) = T_{\mu\nu}^M \quad (2)$$

Where,

$$\square \equiv \nabla^\mu \nabla_\mu, \quad F(R) \equiv \frac{df(R)}{dR} \quad (3)$$

Here, ∇_μ is the covariant derivative and $T_{\mu\nu}$ is the standard matter energy-momentum tensor derived from the Lagrangian L_m .

3. Metric and Field Equations

The Spatially Homogeneous and Anisotropic Bianchi type-III space time is given by

$$ds^2 = dt^2 - A^2 dx^2 - B^2 e^{-2ax} dy^2 - C^2 dz^2 \quad (4)$$

Where A, B, C are cosmic scale factors and are functions of cosmic time t and a is a constant.

The corresponding Ricci Scalar curvature for Bianchi type-III model is given by

$$R = -2 \left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{a^2}{A^2} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} \right] \quad (5)$$

Where, dot represents derivative with respect to t .

The energy-momentum tensor T_{ij} for a Bulk Viscous fluid distribution is given by

$$T_{ij} = (\rho + \bar{p})u_i u_j - \bar{p}g_{ij} \quad (6)$$

Where,

$$\bar{p} = p - \eta u_{;i}^i = p - 3\eta H \quad (7)$$

is the effective pressure, η is the coefficient of bulk viscosity, p is the isotropic pressure, ρ is the energy density, $3\eta H$ is usually known as Bulk Viscous pressure, H is Hubble's parameter, and u^i is fluid four-velocity vector satisfying

$$u_i u^i = 1 \quad (8)$$

Therefore, the dynamics of cosmic evolution does not change fundamentally by the inclusion of viscous term in the energy momentum tensor. In the co-moving co-ordinate system, we have from the above equations

$$T_1^1 = T_2^2 = T_3^3 = -\bar{p}, \quad T_4^4 = \rho, \quad T_i^j = 0, \quad i \neq j \quad (9)$$

Using Eqns. (6)-(9), the corresponding field equations (2) for Bulk Viscous in respect of the Bianchi type-III space-time reduce to the following set of equations

$$\left(\frac{\ddot{A}}{A} - \frac{a^2}{A^2} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} \right) F - \frac{1}{2} f(R) + \ddot{F} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \dot{F} = -\bar{p} \quad (10)$$

$$\left(\frac{\ddot{B}}{B} - \frac{a^2}{A^2} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} \right) F - \frac{1}{2} f(R) + \ddot{F} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \dot{F} = -\bar{p} \quad (11)$$

$$\left(\frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} \right) F - \frac{1}{2} f(R) + \ddot{F} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \dot{F} = -\bar{p} \quad (12)$$

$$\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} \right) F - \frac{1}{2} f(R) + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \dot{F} = \rho \quad (13)$$

$$a \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) F = 0 \quad (14)$$

From Eqn. (14), we get

$$A = c_1 B$$

Where, c_1 is the constant of integration.

Without loss of generality, we can consider $c_1 = 1$ for the sake of simplicity.

Hence we get

$$A = B \quad (15)$$

The conservation equation for energy-momentum $T_{i;j}^i = 0$ gives

$$\dot{\rho} + (\rho + \bar{p}) \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 0 \quad (16)$$

4. Solutions of the Field Equations

Using Eqn. (15) in Eqns. (10)-(13), we get

$$\left(\frac{\ddot{B}}{B} - \frac{a^2}{B^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{B}\dot{C}}{BC} \right) F - \frac{1}{2} f(R) + \ddot{F} + \left(2 \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \dot{F} = -\bar{p} \quad (17)$$

$$\left(\frac{\ddot{C}}{C} + 2 \frac{\dot{B}\dot{C}}{BC} \right) F - \frac{1}{2} f(R) + \ddot{F} + \left(2 \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \dot{F} = -\bar{p} \quad (18)$$

$$\left(2 \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} \right) F - \frac{1}{2} f(R) + \left(2 \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \dot{F} = \rho \quad (19)$$

The physical quantities of observational interest in cosmology are spatial volume V , mean Hubble parameter H , the expansion scalar θ , the mean anisotropy parameter A_m , the shear scalar σ^2 , and the deceleration parameter q . They are defined as:

Spatial Volume

$$V = ABC = B^2 C \quad (20)$$

Mean Hubble Parameter

$$H = \frac{1}{3} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = \frac{1}{3} \left(2 \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \quad (21)$$

Scalar Expansion

$$\theta = 3H \quad (22)$$

Anisotropic Parameter

$$A_m = \frac{1}{3} \sum_{i=1}^3 \left(\frac{\Delta H_i}{H} \right)^2 \quad (23)$$

Where, H_i ($i = 1, 2, 3$) are the directional Hubble parameters in the directions of x , y and z axes respectively. If $A_m = 0$, then the universe expands isotropically. Further, any anisotropic model of the universe with diagonal energy-momentum tensor approaches to isotropy if $A_m \rightarrow 0$, $V \rightarrow +\infty$ and $\rho \rightarrow 0$ as $t \rightarrow +\infty$ ([67],[68]).

Shear Scalar

$$\sigma^2 = \frac{3}{2} A_m H^2 \quad (24)$$

Deceleration Parameter

$$q = \frac{d}{dt} \left(\frac{1}{H} \right) - 1 \quad (25)$$

Here the deceleration parameter q measures the rate of expansion of the universe. The sign of q indicates the state of expanding universe. If $q < 0$ or $q > 0$, respectively, then it represents inflation or deflation of the universe while $q = 0$ shows expansion with constant velocity.

The average scale factor of Bianchi type-III model is defined as

$$a = \left(V^{\frac{1}{3}} \right) \quad (26)$$

Here, we have three independent field equations containing $B, C, \eta, \rho, p, \bar{p}F, f$ eight unknowns, so we shall assume extra conditions to obtain unique

solutions of the field equations. For the complete determination of the exact solutions, additional constraints relating these parameters are required.

For a Barotropic fluid the combined effect of the proper pressure and the Barotropic bulk viscous pressure can be expressed as

$$\bar{p} = p - 3\eta H = \gamma p \quad (27)$$

Where,

$$p = \gamma_0 \rho, \quad 0 \leq \gamma_0 \leq 1 \quad (28)$$

Model I

We assume that solutions of the scale factors of the forms

$$\frac{\dot{B}}{B} = \frac{m_1}{t^n}, \quad \frac{\dot{C}}{C} = \frac{m_2}{t^n} \quad (29)$$

Where, n is a positive constant.

On integration of Eqn. (29), we obtain

$$A = B = a_1 \exp \left\{ \frac{m_1 t^{1-n}}{1-n} \right\} \quad (30)$$

$$C = a_2 \exp \left\{ \frac{m_2 t^{1-n}}{1-n} \right\} \quad (31)$$

Where, a_1, a_2 are constants of integration and $n \neq 1$.

We also assume the result established by Sharif and Shamir [46] in $f(R)$ gravity that shows

$$F \propto a^m \quad (32)$$

Where, m is an arbitrary constant, $a = (V)^{\frac{1}{3}}$ is a scale factor.

Eqn. (32) leads to

$$\begin{aligned} F &= l a^m, \\ F &= l (V)^{\frac{m}{3}}, \\ F &= l (B^2 C)^{\frac{m}{3}} \end{aligned} \quad (33)$$

Where, l is proportionality constant.

Using Eqns. (16), (27) and (29), we obtain

Energy Density

$$\rho = c_2 \exp\left\{\frac{-(1+\gamma)(2m_1+m_2)t^{1-n}}{1-n}\right\} \quad (34)$$

Using Eqns. (28) and (34), we get

Isotropic pressure

$$p = \gamma_0 c_2 \exp\left\{\frac{-(1+\gamma)(2m_1+m_2)t^{1-n}}{1-n}\right\}$$

Effective pressure

$$\bar{p} = \gamma c_2 \exp\left\{\frac{-(1+\gamma)(2m_1+m_2)t^{1-n}}{1-n}\right\} \quad (35)$$

Using Eqns. (7), (34) and (35), we have

$$R = -2 \left\{ \frac{(3m_1^2 + m_2^2 + 2m_1m_2)}{t^{2n}} - \frac{(2m_1n + m_2n)}{t^{n+1}} - \frac{a^2}{a_1^2 \exp\left[\frac{2m_1t^{1-n}}{1-n}\right]} \right\} \quad (38)$$

$f(R)$ turns out to be

$$\begin{aligned} \frac{1}{2} f(R) = & \left(\frac{2m_1^2}{t^{2n}} - \frac{2m_1n}{t^{n+1}} + \frac{m_2^2}{t^{2n}} - \frac{m_2n}{t^{n+1}} \right) l \left\{ a_1^2 a_2 \exp\left[\frac{(2m_1+m_2)t^{1-n}}{1-n}\right] \right\}^{\frac{m}{3}} \\ & + \left(\frac{2m_1+m_2}{t^n} \right) \left\{ l a_1^2 a_2 \exp\left[\frac{(2m_1+m_2)t^{1-n}}{1-n}\right] (2m_1+m_2)t^{-n} \left[a_1^2 a_2 \exp\left[\frac{(2m_1+m_2)t^{1-n}}{1-n}\right] \right]^{\frac{m-1}{3}} \right\} \\ & - c_2 \exp\left\{\frac{-(1+\gamma)(2m_1+m_2)t^{1-n}}{1-n}\right\} \end{aligned} \quad (39)$$

The mean Hubble parameter is given by

$$H = \frac{1}{3} \frac{(2m_1+m_2)}{t^n} \quad (40)$$

The spatial volume of the universe is given by

$$V = \left\{ a_1^2 a_2 \exp\left[\frac{(2m_1+m_2)t^{1-n}}{1-n}\right] \right\} \quad (41)$$

The expansion scalar is given by

Coefficient of Bulk Viscosity

$$\eta = \frac{1}{3H} (\gamma_0 - \gamma) c_2 \exp\left\{\frac{-(1+\gamma)(2m_1+m_2)t^{1-n}}{1-n}\right\} \quad (36)$$

Using Eqns. (30), (31) and (33), we obtain

$$F = l \left\{ a_1^2 a_2 \exp\left[\frac{(2m_1+m_2)t^{1-n}}{1-n}\right] \right\}^{\frac{m}{3}} \quad (37)$$

Ricci scalar curvature can be found out as

$$\theta = \frac{(2m_1+m_2)}{t^n} \quad (42)$$

The mean anisotropic parameter is

$$A_m = \frac{2(m_1-m_2)^2}{(2m_1+m_2)^2} \quad (43)$$

The shear scalar is given by

$$\sigma^2 = \frac{1}{3} \frac{(m_1-m_2)^2}{t^{2n}} \quad (44)$$

The deceleration parameter is given by

$$q = \frac{3nt^{n-1}}{2m_1 + m_2} - 1 \quad (45)$$

Clearly q is positive for $t > \left(\frac{2m_1 + m_2}{3n}\right)^{\frac{1}{n-1}}$ and is negative for $t < \left(\frac{2m_1 + m_2}{3n}\right)^{\frac{1}{n-1}}$. The deceleration parameter indeed has a sign flip at $t = \left(\frac{2m_1 + m_2}{3n}\right)^{\frac{1}{n-1}}$. For $t < \left(\frac{2m_1 + m_2}{3n}\right)^{\frac{1}{n-1}}$, the solution gives an accelerating model of the universe. When $t > \left(\frac{2m_1 + m_2}{3n}\right)^{\frac{1}{n-1}}$, our solution represents a decelerating model of the universe. Thus our theoretical models are consistent with the results of recent observations.

We observe that the spatial volume is constant at $t = 0$. At this epoch, the energy density ρ is finite and θ, σ are zero. For $0 < t < \infty$, the physical parameters $\theta, \sigma, p, \rho, \eta$ are well behaved and are decreasing functions of time. As $t \rightarrow \infty$, the spatial volume tends to infinity if $n < 1$ and the physical parameters tend to zero. Thus, for physical reality of the model, we must have $0 < n < 1$. The model essentially gives an empty space-time as large time.

Also we find that $\frac{\sigma}{\theta}$ tends to a constant limit as $t \rightarrow \infty$, which shows that the anisotropy in the universe is maintained throughout.

Model II

We now obtain solution of the field equations (17) to (19) for $n = 1$.

For $n = 1$, the scale factors in Eqn. (29) are given by

$$\frac{\dot{B}}{B} = \frac{m_1}{t}, \quad \frac{\dot{C}}{C} = \frac{m_2}{t} \quad (46)$$

On integration of Eqn. (46), we obtain

$$A = B = k_1 t^{m_1} \quad (47)$$

$$C = k_2 t^{m_2} \quad (48)$$

Where, k_1, k_2 are constants of integration.

Using Eqns. (16), (27) and (46), we obtain

Energy Density

$$\rho = k_3 t^{-(1+\gamma)(2m_1+m_2)} \quad (49)$$

Where, k_3 is a constant of integration.

Using Eqns. (28) and (49), we get

Isotropic pressure

$$p = \gamma_0 k_3 t^{-(1+\gamma)(2m_1+m_2)} \quad (50)$$

Effective pressure

$$\bar{p} = \gamma k_3 t^{-(1+\gamma)(2m_1+m_2)} \quad (51)$$

Using Eqns. (7), (50) and (51), we have

Coefficient of Bulk Viscosity

$$\eta = \frac{1}{3H} (\gamma_0 - \gamma) k_3 t^{-(1+\gamma)(2m_1+m_2)} \quad (52)$$

The bulk viscosity coefficient is a decreasing function of time. The effect of bulk viscosity is to produce a change in perfect fluid and hence exhibit essential influence on the character of its solution. The effect is clearly visible in isotropic pressure and energy.

Using Eqns. (47), (48) and (33), we obtain

$$F = l \left\{ k_1^2 k_2 t^{(2m_1+m_2)} \right\}^{\frac{m}{3}} \quad (53)$$

Ricci scalar curvature can be found out as

$$R = -2 \left\{ \frac{(m_1^2 + m_2^2 + 2m_1 m_2 - 2m_1 - m_2)}{t^2} - \frac{a^2}{k_1^2 t^{2m_1}} \right\} \quad (54)$$

$f(R)$ turns out to be

$$\begin{aligned} \frac{1}{2} f(R) = & \left(\frac{2m_1(m_1-1)}{t^2} + \frac{m_2(m_2-1)}{t^2} \right) l \left\{ k_1^2 k_2 t^{(2m_1+m_2)} \right\}^{\frac{m}{3}} \\ & + \left(\frac{2m_1+m_2}{t^2} \right) \left\{ \frac{l k_1^2 k_2 m}{3} (2m_1+m_2) \left[k_1^2 k_2 t^{(2m_1+m_2)} \right]^{\frac{m}{3}-1} t^{2m_1+m_2-1} \right\} \\ & - k_3 t^{-(1+\gamma)(2m_1+m_2)} \end{aligned} \quad (55)$$

The mean Hubble parameter is given by

$$H = \frac{1}{3} \frac{(2m_1 + m_2)}{t} \quad (56)$$

The spatial volume of the universe is given by

$$V = \left\{ k_1^2 k_2 t^{(2m_1+m_2)} \right\}^{\frac{1}{3}} \quad (57)$$

The expansion scalar is given by

$$\theta = \frac{(2m_1 + m_2)}{t} \quad (58)$$

The mean anisotropic parameter is

$$A_m = \frac{2(m_1 - m_2)^2}{(2m_1 + m_2)^2} \quad (59)$$

The shear scalar is given by

$$\sigma^2 = \frac{1}{3} \frac{(m_1 - m_2)^2}{t^2} \quad (60)$$

The deceleration parameter is given by

$$q = \frac{3}{2m_1 + m_2} - 1 \quad (61)$$

From Eqn. (61), we observe that

$$q < 0 \text{ if } 2m_1 + m_2 > 3$$

and

$$q > 0 \text{ if } 2m_1 + m_2 < 3$$

Thus, our solution represents an accelerating model of the universe if $2m_1 + m_2 > 3$ and decelerating model if $2m_1 + m_2 < 3$. Thus our theoretical models are consistent with the results of recent observations.

The spatial volume is zero at $t=0$. At this epoch, the energy density ρ is infinite and θ, σ

are also infinite. Thus the model starts with a big-bang singularity at $t=0$. The above parameters decrease with passage of time. The spatial volume increases as time increases and becomes infinite at late time. As $t \rightarrow \infty$, the physical parameters $\theta, \sigma, p, \rho, \eta$ tend to zero. Thus, the model represents an expanding shearing and non-rotating universe, which essentially gives an empty for large time.

Also we find that $\frac{\sigma}{\theta}$ tends to zero as $t \rightarrow \infty$,

which shows that the anisotropy in the universe is maintained throughout.

Model III

The two observable parameters q and H are related by the relation

$$q = -1 + \frac{d}{dt} \left(\frac{1}{H} \right) \quad (62)$$

This equation, on integration, gives the scale factor $a(t)$ as

$$a(t) = e^\delta \exp \int \frac{dt}{(1+q)dt + r} \quad (63)$$

Where, r, δ are arbitrary constants of integration.

For a possible integration of Eqn. (63), we observe that i) $q = \text{constant}$ is an easy choice $a(t)$ as an explicit function of t , as has also been discussed by Bermann [69]. ii) When q is taken to vary with time, an explicit determination of $a(t)$.

For the complete determination of $a(t)$, Abdusattar and Prajapati [70] proposed the following choice of q as

$$q = -\frac{\alpha}{t^2} + (\beta - 1) \quad (64)$$

Where, $\alpha > 0$ is a parameter having the dimension of square of time and $\beta > 1$ is a dimensionless constant. Obviously, the different values of α and β will give rise to different models.

With q given by (64) and $\delta = 0$, $r = 0$, Eqn. (63) can be integrated to give the scale factor as

$$a(t) = \left(t^2 + \frac{\alpha}{\beta} \right)^{\frac{1}{2\beta}} \quad (65)$$

In order to solve the system completely, we assume that $C = V^b$, where b is any constant number.

Then from Eqns. (15), (20) and (65), we obtain the exact expression for the scale factors

$$A = B = \left(t^2 + \frac{\alpha}{\beta} \right)^{\frac{3-3b}{4\beta}} \quad (66)$$

$$C = \left(t^2 + \frac{\alpha}{\beta} \right)^{\frac{3b}{2\beta}} \quad (67)$$

Using Eqns. (33), (65), we obtain

$$F = l \left(t^2 + \frac{\alpha}{\beta} \right)^{\frac{m}{2\beta}} \quad (68)$$

Using Eqns. (16), (27), (66), (67), we get

$$R = -2 \left\{ \left(\frac{3-3b}{2\beta} \right) \frac{1}{\left(t^2 + \frac{\alpha}{\beta} \right)} \left[\left(\frac{3-3b-4\beta}{2\beta} \right) \frac{t^2}{\left(t^2 + \frac{\alpha}{\beta} \right)} + 1 \right] \right\} + \frac{3b}{\beta} \frac{1}{\left(t^2 + \frac{\alpha}{\beta} \right)} \left[\left(\frac{3b-2\beta}{\beta} \right) \frac{t^2}{\left(t^2 + \frac{\alpha}{\beta} \right)} + 1 \right] \quad (73)$$

$$- \frac{a^2}{\left(t^2 + \frac{\alpha}{\beta} \right)^{\frac{3-3b}{2\beta}}} + \left(\frac{3-3b}{2\beta} \right)^2 \frac{t^2}{\left(t^2 + \frac{\alpha}{\beta} \right)^2} + 2 \left(\frac{3-3b}{2\beta} \right) \frac{3b}{\beta} \frac{t^2}{\left(t^2 + \frac{\alpha}{\beta} \right)^2}$$

$f(R)$ turns out to be

$$\rho = c_3 \left(t^2 + \frac{\alpha}{\beta} \right)^{\frac{-3(1+\gamma)}{2\beta}} \quad (69)$$

Where, c_3 is constant of integration.

Using Eqns. (27), (69), we have

$$\bar{p} = \gamma c_3 \left(t^2 + \frac{\alpha}{\beta} \right)^{\frac{-3(1+\gamma)}{2\beta}} \quad (70)$$

Using Eqns. (28), (69), we have

$$p = \gamma_0 c_3 \left(t^2 + \frac{\alpha}{\beta} \right)^{\frac{-3(1+\gamma)}{2\beta}} \quad (71)$$

Using Eqns. (7), (70) and (71), we have

Coefficient of Bulk Viscosity

$$\eta = \frac{1}{3H} (\gamma_0 - \gamma) c_3 \left(t^2 + \frac{\alpha}{\beta} \right)^{\frac{-3(1+\gamma)}{2\beta}} \quad (72)$$

Ricci scalar curvature can be found out as

$$\begin{aligned} \frac{1}{2} f(R) = & \left(\left\{ 2 \left(\frac{3-3b}{2\beta} \right) \frac{1}{\left(t^2 + \frac{\alpha}{\beta} \right)} \left[\left(\frac{3-3b-4\beta}{2\beta} \right) \frac{t^2}{\left(t^2 + \frac{\alpha}{\beta} \right)} + 1 \right] + \frac{3b}{\beta} \frac{1}{\left(t^2 + \frac{\alpha}{\beta} \right)} \left[\left(\frac{3b-2\beta}{\beta} \right) \frac{t^2}{\left(t^2 + \frac{\alpha}{\beta} \right)} + 1 \right] \right\} \right. \\ & \left. + \left\{ 2 \left(\frac{3-3b}{2\beta} \right) \frac{t}{\left(t^2 + \frac{\alpha}{\beta} \right)} + \frac{3b}{\beta} \frac{t}{\left(t^2 + \frac{\alpha}{\beta} \right)} \right\} \frac{lm t}{\beta} \left(t^2 + \frac{\alpha}{\beta} \right)^{\frac{m-2\beta}{2\beta}} - c_3 \left(t^2 + \frac{\alpha}{\beta} \right)^{\frac{-3(1+\gamma)}{2\beta}} \right) \left(t^2 + \frac{\alpha}{\beta} \right)^{\frac{m}{2\beta}} \end{aligned} \quad (74)$$

The mean Hubble parameter is given by

$$H = \frac{t}{\beta \left(t^2 + \frac{\alpha}{\beta} \right)} \quad (75)$$

The spatial volume of the universe is given by

$$V = \left(t^2 + \frac{\alpha}{\beta} \right)^{\frac{3}{2\beta}} \quad (76)$$

The expansion scalar is given by

$$\theta = \frac{3t}{\beta \left(t^2 + \frac{\alpha}{\beta} \right)} \quad (77)$$

The mean anisotropic parameter is

$$A_m = \frac{(3b-1)^2}{2} \quad (78)$$

The shear scalar is given by

$$\sigma^2 = \frac{3(3b-1)^2}{4\beta^2} \frac{t^2}{\left(t^2 + \frac{\alpha}{\beta} \right)^2} \quad (79)$$

The deceleration parameter is given by

$$q = \frac{\beta \left(t^2 - \frac{\alpha}{\beta} \right)}{t^2} - 1 \quad (80)$$

At the origin of the time coordinate $t=0$, we find that $a(0) \neq 0$, $\dot{a}(0) = 0$ but $\ddot{a}(0) = \text{constant}$, which indicate that the model is free from initial singularity and start expanding with finite acceleration. From Eqn. (64), we observe that the deceleration parameter $q \rightarrow -\infty$ at $t=0$ and

reduces to zero at $t = \sqrt{\frac{\alpha}{\beta-1}}$. The period of

accelerated expansion also depends on the values of α and β . Afterwards, the model decelerates with deceleration parameter q approaching $\beta-1$ for sufficiently large values of t . For $\beta=1$, the Universe has an accelerated expansion throughout the evolution. We see that θ, σ^2, H ultimately tend to zero as $t \rightarrow \infty$.

5. Conclusion

In this paper, we have obtained spatial and homogenous Bianchi type-III cosmological models with bulk viscous as a matter in $f(R)$ gravity. We have presented two classes of physically viable cosmological models for $n \neq 1$ and $n=1$. The first model essentially gives an empty space-time as

large time. For $t < \left(\frac{2m_1 + m_2}{3n} \right)^{\frac{1}{n-1}}$, the solution gives an accelerating model of the universe.

When $t > \left(\frac{2m_1 + m_2}{3n} \right)^{\frac{1}{n-1}}$, our solution represents a

decelerating model of the universe. For $n \neq 1$, the model evolves with a finite volume at $t=0$ and does not approach isotropy as $t \rightarrow \infty$. For large time, the energy density becomes zero. The second model represents an expanding shearing and non-rotating universe, which essentially gives an empty

space for large time. For, $n = 1$ the model starts evolving with a big-bang singularity at $t = 0$. This model represents an accelerating or decelerating universe according as $t = 2m_1 + m_2$ which is greater than 3 or less than 3. The anisotropy is maintained throughout in the models. The first and second models essentially give empty space for large time and are consistent with the recent observations on cosmology. In both models, Bulk Viscosity, decreases as t increases, which is in accordance with the well-known fact that Bulk Viscosity decreases with time and leads to inflationary model [53]. The third model is free from initial singularity and start expanding with finite acceleration. An important aspect of the third model is that it starts from rest with a finite volume and finite acceleration, which then gradually decrease and reduce to zero after some time. Thereafter, the model decelerates with gradually increasing deceleration approaching a constant value for large values of t . It is also observed that all the physical parameters are decreasing functions of time and they approach zero for large t . Thus there is an obvious provision for the smoothness in the universe during the accelerated phase of the expansion and for the structure formation in the universe during the decelerated phase of the expansion. The models obtained may throw some light on our understanding of $f(R)$ cosmology.

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