

Quiver Gauge Theories from Lie Superalgebras

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We discuss quiver gauge models with matter fields based on Dynkin diagrams of Lie superalgebra structures. We focus on $A(1,0)$ case and find first that it can be related to intersecting complex cycles with genus g . Using toric geometry, $A(1,0)$ quivers are analyzed in some details and it is shown that $A(1,0)$ can be used to incorporate fundamental fields to a product of two unitary factor groups. We expect that this approach can be applied to other kinds of Lie superalgebras.

Four dimensional gauge theories have attracted substantial attention in connection with supergravity models in higher dimensional theories. In particular, they appears in the study of D-brane world volume gauge theories embedded in such theories compactified either on Calabi-Yau manifolds or G_2 manifolds. The group and matter content of the resulting models are obtained from the ADE singularities of K3 fibers and a non-trivial geometry describing the base space of internal manifolds, respectively [1-7]. In this way, the complete set of physical parameters of the gauge theory is related to the moduli space of the associated manifolds. This program is called geometric engineering allowing one to have exact solutions for the Coulomb branch moduli space [2]. A nice way to encode the physical information of such gauge theories is to use the quiver approach [8,9]. The corresponding gauge models are usually called quiver gauge theories. For a gauge theory with gauge symmetry given by the product

$$G = \prod_i G_i, \quad (1)$$

we represent its physical content by a quiver, or equivalently a graph, formed by nodes and links. For each node, one associates a gauge factor G_i (i denotes the number of nodes). Links between two nodes are associates with a_{ij} matter transforming in bi-fundamental representation of two factor of the gauge group $G_i \times G_j$.

Recently, many examples of quivers have been built in connection with string theory moving on manifolds with special holonomy groups. In particular, such quivers have been based on toric graphs and Dynkin diagrams. The last class is the most studied one due to its relation with the physics of D-branes placed near singularities classified by Cartan matrices of Lie algebras. In this way, the physical content can be encoded in quivers sharing the same properties like Dynkin diagrams. A close inspection reveals that one may have three models based on Lie algebras classification with symmetric Cartan matrices [4,5]. The classification of Dynkin diagrams led to the following three different quiver gauge theories:

1. Ordinary ADE quiver gauge theories
2. Affine ADE quiver gauge theories
3. Indefinite quiver gauge theories.

In string and related theories, the models of first class are not conformally invariant [2]. They can be obtained from D-branes wrapping a collection of intersecting cycles according to ordinary ADE Dynkin diagrams of finite Lie algebras. The second class of models are conformal gauge theories involving a remarkable realization in terms of D-branes wrapping elliptic singularities [2,3]. The last one is the poor one from the D-brane realization point of view. There are a few models based on such Lie algebras including hyperbolic cases [4,5]. It is worth noting that the last two models can be considered as possible extension of ordinary ADE quiver gauge theories. The

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derivation of such models is based on the usual philosophy where one uses in the building of the Dynkin diagrams from the finite ones by adding extra nodes. In the quiver method, this can be understood by adding more gauge factors and matter fields. More precisely, The extra nodes allow one to implement new physical constraints on gauge factors and matter field contributions. In particular, the affine node has been explored to engineer geometrically conformal models which have a remarkable realization as D-brane world volume gauge theories obtained from elliptic singularities.

On the basis of this classification of quiver gauge theories, it is natural to think about quiver gauge theories based on other algebra structures. In this paper, we are interested in Lie superalgebras. Such symmetries have been extensively studied in various contexts [10-15]. They can be thought of as a special extension of bosonic Lie symmetries. Using quiver method, such symmetries can be explored to build a new class of quiver gauge theories. This may give an explicit evidence for the role played by Lie superalgebras string theory and related models. It may also open an issue to look for the corresponding singularities in the compactification mechanism including superCalabi-Yau manifolds.

Roughly speaking, there are many examples of such Lie algebras. In this paper, we are interested in the case of $A(1,0)$, which can be thought of as a particular extension of A_1 bosonic Lie algebras. This example has been shown to deal with several aspects of (supersymmetric) integrable conformal models [16,17]. In particular, it is relevant to the classification program of extended $\mathcal{N} = 2$ superconformal algebras and string theories obtained by gauging $\mathcal{N} = 2$ Wess-Zumino-Witten models. The choice of $A(1,0)$ is motivated from the fact that A_1 is considered as the building block of the ADE classification of simply laced Lie algebras. Indeed, since the algebra $A(1,0)$ is the simplest extension of Lie superalgebras associated with the ADE classification, one can shortly review its structure. This structure is quite the same as the case of A_2 but with some novelties. Let us give briefly how this symmetry can be built from the underlying bosonic one. More details can be found in [18]. Indeed, $A(1,0)$ is an eight (4+4) dimensional algebra with rank 2. From the fundamental 3-dimensional representation of $A(1,0)$, one can write down the relations obeyed by its four bosonic generators and its four fermionic generators. Its root system has

been studied extensively involving fermionic and bosonic roots. In fact, it has been shown that this algebra has two different root systems. The first root system involves two fermionic nonzero roots α_1, α_2 having length square zero, and a normalized bosonic nonzero root with length square 2 given by $(\alpha_1 + \alpha_2)$. The corresponding Cartan matrix reads as

$$K_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2)$$

and it is associated with the following Dynkin diagram

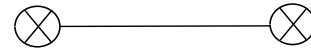


FIG. 1: Dynkin diagram of $A(1,0)$ associated with two fermionic simple roots.

The second possibility contains one fermionic simple nonzero root α_1 having length square zero and a simple bosonic root α_2 with length square 2. The normalized fermionic one nonzero root with length square 0 is $(\alpha_2 - \alpha_1)$. The total root system is given by $\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_2 - \alpha_1)\}$. The corresponding Cartan matrix takes the form

$$K_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \quad (3)$$

This matrix reproduces the following Dynkin Diagram



FIG. 2: Dynkin diagram of $A(1,0)$ associated with one fermionic simple root and one bosonic simple root.

Let us now summarize the above discussion as:

- First, we observe that $A(1,0)$ is realized by two different Cartan matrices leading to two different Dynkin diagrams. One of them is purely fermionic.
- Second, these matrices are different from the one associated with the bosonic A_2 Lie algebras given by

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad (4)$$

Based on the correspondence between quiver gauge theories and Dynkin diagrams, we would like to construct quivers based on Lie superalgebras. This may offer a new approach to deal with such algebras from string theory point of view using geometric engineering method. As usual, for each Dynkin diagram of $A(1,0)$, one associates a quiver model that can be obtained from D-branes wrapping appropriate cycles either in Calabi-Yau manifolds or orbifolds spaces. The corresponding quiver gauge models constitute a special class based on bosonic and fermionic nodes.

To give a quiver gauge theory associated with the above Cartan matrices, it is interesting first to find a geometric meaning of the corresponding roots in homology of Calabi-Yau Compactifications. Inspired from bosonic quiver gauge theories, the above roots should have a possible connection with middle cohomology groups in internal spaces. Based on geometric engineering method in II superstrings, the roots corresponding to the above Cartan matrices may produce a particular set of intersecting cycles embedded in two dimensional Calabi-Yau manifolds including the K3 surfaces and ALE spaces.

A close inspection in intersecting theory reveals that a possible candidate can be given in terms of two intersecting homological two cycles. This observation may help us to get information on the complex two dimensional homology of the possible candidate for the geometry associated with the Cartan matrices of $A(1,0)$ Lie superalgebra. A priori, there are many ways one may be able to give a consistent interpretation. However, the way we follow here will be based on the nice interplay between toric geometry of the ADE surfaces and Lie algebras [2]. Indeed, the connection we are after requires the introduction of complex cycles with genus g . In fact, we will associate a simple root with a Riemann surface of genus g that substitutes the $\mathbb{CP}^1$ sphere appearing in the bosonic ADE classification. To get our formulation, we use the following points:

1. The supercommutator of Z_2 graded algebra satisfies the relation

$$[x, y] = -(-1)^{|x||y|}[y, x]$$

where $|x|$ denotes the degree of x either 0 or 1.

2. The study of complex curves on two dimensional complex surfaces.

Based on these points, we can associate a set of complex curves to a root system according to the following rules:

1. To each bosonic root associate a complex curve of zero genus
2. To each fermionic root associate a complex curve with genus one.

These values of genus ($g = 0, 1$) are inspired from the Z_2 graded structure. With these ingredients, the Cartan matrices of $A(1,0)$ can be related to a complex surface with rank 2 cohomology space $H^2(X, \mathbb{Z})$. The latter is generated by two cycle C_i genus g_i . More specifically, we propose the intersection form

$$C_i C_j = -2\delta_{ij}(g_i - 1) + (-1)^{(g_i + g_j)}(1 - \delta_{ij})C_j C_i \quad (5)$$

Where, $g_i = 0, 1$. This formulation has the following nice features. First, these intersection numbers reproduce the elements of the above Cartan matrices. Second, for $g_i = 0$, we recover the bosonic case with rank 2 given in Eqn. (4). This formulation may be considered as a complete picture for the above three Cartan matrices.

It is worth noting that this formulation is adaptable to all simply laced superalgebras including $A(m, n)$ cases. We anticipate that the evaluation of non simply laced in general will not be easy. This will be addressed elsewhere where we also intend to discuss the implementation of complex curves with higher genus. It would therefore be of interest to try to extract information on graded structure based on higher order discrete groups. We believe that this observation deserves to be studied further.

As already mentioned, all what we know about quiver gauge theories can be extended to models based on Lie superalgebras. Mimicking the analysis one has done for the above classification models, we can get a D-brane realization dealing with quivers based on Lie superalgebras. The general study is beyond the scope of the present work, though we will consider two explicit examples relying on the above Cartan matrices. The gauge symmetry of such quivers involves naturally two factors $U(N_1) \times U(N_2)$ with running gauge coupling constants λ_1, λ_2 , respectively. Each factor is engineered on a node of the Dynkin diagrams of $A(1,0)$ Lie superalgebra. The corresponding models are quite similar to the ones built on the ordinary A_2 . More precisely, they can be viewed as a deformation of A_2 quiver gauge theory studied in [2]. The only difference

will appear in the computation of the beta function. In $N = 2$ supersymmetric models in four dimensions [19,20], the holomorphic beta functions β_i depend linearly on the ranks of the gauge factors where it has been shown to be proportional to Cartan matrices. Taking into account this observation, the calculation can be extended to Lie superalgebras, namely

$$\beta_i \sim K_{ij}(g_i, g_j) N_j \quad (6)$$

Where, now $K_{ij}(g_i, g_j)$ are Cartan matrices of the above Lie superalgebras which can be reproduced from Eqn. (5). In $N = 2$ quivers, the vanishing condition of such functions is intimately related to the conformal invariance [19,20]. This condition has a nice interpretation in toric geometry realization of Calabi-Yau manifolds [21,22]. In such a framework, Eqn. (6) can be reinterpreted as a relation between toric vertices defining the toric manifolds on which the quiver is built. It has been observed that the conformal condition can be reached by considering the following general relation

$$K_{ij}(g_i, g_j) N_j - M_i = 0 \quad (7)$$

Where, M_i are fundamental matter fields required by the Calabi-Yau constraint. It turns out that there are many ways to solve such equations. As the bosonic case, a possible way forces us to implement auxiliary nodes producing quiver with more than two gauge factors. This procedure allows one to modify the above Cartan matrices leading to a Calabi-Yau geometry. However, to get the desired quiver the extra nodes behave like non dynamic gauge factor and they should be associated with matter fields. In fact, these models can be obtained by assuming that auxiliary nodes represent non compact cycles in toric geometry realization of Calabi-Yau manifolds [2].

Let us give concrete examples. First we consider the case where $g_i = 1$ associated with Eqn. (2). Indeed, mimicking the analysis made for bosonic ADE Lie algebras, the Calabi-Yau condition requires the following modified Cartan matrix

$$\widetilde{K}_{il}(g_i = 1) = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \quad (8)$$

This matrix can have a nice physical interpretation in terms of $N = 2$ linear sigma models describing the conformal field theory on Calabi-Yau string background, required by $\sum_i \widetilde{K}_{il} = 0$. This

target string background can be analyzed by solving the D-term potential dealing with a gauge group $U(1)^2$ with 4 matter fields ψ_i

$$\begin{aligned} -|\psi_1|^2 + |\psi_3|^2 &= r_1 \\ |\psi_2|^2 - |\psi_4|^2 &= r_2 \end{aligned} \quad (9)$$

Where the $r_{1,2}$ are FI coupling parameters. In this sigma model realization, the $U(1)^2$ gauge symmetry can be identified with the Cartan subalgebras of $A(1,0)$ Lie superalgebra. Eqn. (9) has a nice toric geometry relation given by

$$\widetilde{K}_{il} v_\ell = 0 \quad (10)$$

Where v_ℓ toric vertices are given by $v_\ell = (N_\ell, \star, \star)$ required by the Calabi-Yau conditions. In this way, N_ℓ can be identified with gauge group ranks. However, the other components are spectator integers playing no role in quiver models.

It turns out that Eqn.(6) may be thought of as a particular situation of Eqn. (10). Inspired by the trivalent vertex method used in conformal field theory, the above toric equations can be interpreted as quiver equations associated with fundamental matter. Assuming that the gauge couplings associated with the external nodes go to zero, the corresponding quiver is described by $U(N) \times U(N)$ gauge symmetry and $U(N) \times U(N)$ flavor group. This quiver data may be obtained from D-branes wrapped on cycles with large volumes. This model is encoded in Fig. 3 where the external nodes are associated with $U(N) \times U(N)$ flavor group.

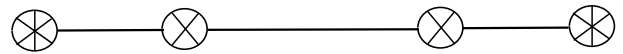


FIG. 3: The middle nodes are associated with $U(N) \times U(N)$ gauge symmetry while the external nodes are associated with fundamental matter

To get the quiver associated the second Dynkin diagram, one may follow the approach used for the first example. Taking the corresponding modified Cartan matrix

$$\widetilde{K}_{il}(g_1 = 1, g_2 = 0) = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 \end{pmatrix} \quad (11)$$

required by conformal invariance in toric geometry language, we derive the quiver model associated with the Cartan matrix given in Eqn. (3).

This can be obtained by considering the vanishing limits of the gauge coupling constants associated with two external nodes corresponding to extra lines and columns of the matrix (11). Equivalently, we take the extra cycles to be very large. The dynamics associated with such cycles become weak leading to a spectator symmetry group. This assumption produces a quiver theory with $U(N) \times U(N)$ gauge symmetry and flavor group of type $U(N) \times U(N)$ engineered on the two external nodes.

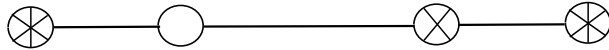


FIG. 4: Quiver model based on the second Cartan matrix of $A(1,0)$.

In the end of this analysis, it is interesting to note the following crucial points:

1. $A(1,0)$ Lie superalgebra encodes only a quiver gauge model even it involves two Dynkin diagrams. We do not understand this coincidence on the physical content of $A(1,0)$ Dynkin diagrams. We believe that there could possibly be a deeper reason behind such a feature. This observation needs to be developed in future.
2. This analysis may be extended to the case of ADE superalgebras. In this way, one has nodes with more than two links. This involves polyvalent vertices in toric geometry representation of Calabi-Yau manifolds. More details on such geometry can be found in [2]. In all geometries, the conformal condition may be considered as a particular expression of toric geometry equations.
3. We expect that the approach is adaptable to a broad variety of geometries represented by supermanifolds constructed as fermionic extensions of toric Calabi-Yau varieties. These geometries can be realized as target spaces of supersymmetric linear sigma models with chiral and fermionic superfields with charges satisfying the super Calabi-Yau condition. A special class of such geometries with ADE singularities have been discussed in [23]. It should be interesting to make contact with these activities.

As we have seen our work comes up with many questions. We shall try to address such issues elsewhere.

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Received: 25 September, 2014

Accepted: 02 October, 2014