

Dirac Equation for Tietz Potential with Coulomb-like and Yukawa-like Tensor Interaction

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It is known that a tensor interaction in the Dirac equation removes the degeneracy in spin and pseudo-spin doublets and therefore leads to conformity with experimental results and data. In this paper, we use the Coulomb and Yukawa tensor interactions to obtain bound state solutions of the Dirac equation for Tietz potential using the Nikiforov-Uvarov method.

1. Introduction

Ginocchio et al. [1-4] established the relationship between the pseudo-spin and the Dirac equation by recognizing that pseudo-orbital angular momentum $\tilde{l} = l + 1$ is nothing but the usual orbital angular momentum l of the lower component of the Dirac spinor, and he went further to show that within the framework of the Dirac theory, the spin symmetry occurs when the difference between the repulsive Lorentz vector potential $V(r)$ and attractive Lorentz scalar potential $S(r)$ is a constant, that is, $\Delta(r) = V(r) - S(r) = \text{const.}$ On the other hand, the pseudo-spin symmetry arises when the sum of repulsive Lorentz vector potential $V(r)$ and attractive Lorentz scalar potential $S(r)$ is a constant, that is, $\Sigma(r) = V(r) + S(r) = \text{const.}$ The concept of pseudo-spin and spin symmetries on a number of potentials have been investigated by many researchers in this field. These include Manning-Rosen potential [5], Eckart potential [6], Hylleraas potential [7], Deng-Fan potential [8], Mobius square potential [9], Tietz potential [10], and Hyperbolic potentials [11-13]. The spin and pseudo-spin symmetries under various phenomenological potentials have been investigated using various methods such as, Nikiforov-Uvarov (NU) method [14], supersymmetric quantum mechanics (SUSYQM) [15], and many others [16]. In most of these studies, due to the mathematical structure of the problem, the tensor is considered as Coulomb-like [17-18] or the Cornell interaction. Hassanabadi et al. were the

first to introduce the Yukawa tensor interaction [19]. The equivalence of the deformed modified Rosen-Morse potential energy model and the Tietz potential energy model has recently been investigated by Sun et al. [20]. Many authors have investigated some typical empirical and analytical potential functions on their equivalence [21-22]. The empirical potential functions include the Rosen-Morse [23], Manning-Rosen potential [24], Tietz potential [25] and Schioberg potential [26]. However, the Tietz potential model has been found to be in good agreement with the experimental Rydberg-Klein-edRees (RKR) data points and *ab initio* calculations [20,27-28]. The well-known Tietz potential for diatomic molecules [20,29] is defined as

$$V(r) = D_e + D_e \frac{(a+b)e^{-4\alpha r} - be^{-2\alpha r}}{(1 - e^{-2\alpha r})^2} \quad (1)$$

Where, D_e is the dissociation energy and a, b and α are three independent parameters.

In the present study, we obtain approximate analytical solutions of the Dirac equation for scalar and vector Tietz potentials together with Coulomb-like and Yukawa-like tensor potentials within the framework of spin and pseudo-spin symmetry limits.

The paper is organized as follows. In Sec. 2, we review the NU method. Sec. 3 is devoted to the Dirac equation for spin and pseudo-spin symmetries. We present the solutions of the Dirac equation under Coulomb tensor interaction in Sec. 4. Solutions of the Dirac equation under Yukawa tensor interaction are presented in Sec. 5 and Sec.

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6. Special case of the potential is discussed in Sec. 7. Finally, we give a brief conclusion in Sec. 8.

2. Methods

2.1. The Nikiforov-Uvarov method

The NU method can solve a second-order differential equation of the form [14]

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0 \quad (2)$$

Where, $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and more direct, we introduce a more compact presentation of the idea. In order to do this, we rewrite Eqn. (1) as follows [29]

$$\begin{aligned} \psi_n''(s) + \left(\frac{c_1 - c_2 s}{s(1 - c_3 s)} \right) \psi_n'(s) \\ + \left(\frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{s^2(1 - c_3 s)^2} \right) \psi_n(s) = 0 \end{aligned} \quad (3)$$

Where,

$$\psi_n(s) = \phi(s)y_n(s) \quad (4)$$

Comparing Eqn. (2) with Eqn. (3), we obtain the following identifications

$$\begin{aligned} \tilde{\tau}(s) &= c_1 - c_2 s \\ \sigma(s) &= s(1 - c_3 s) \\ \tilde{\sigma}(s) &= -\xi_1 s^2 + \xi_2 s - \xi_3 \end{aligned} \quad (5)$$

Following the NU method [14,30], we obtain the following required parameters:

(i) Relevant constants

$$\begin{aligned} c_4 &= \frac{1}{2}(1 - c_1) \\ c_5 &= \frac{1}{2}(c_2 - 2c_3) \\ c_6 &= c_5^2 + \xi_1 \end{aligned}$$

$$\begin{aligned} c_7 &= 2c_4 c_5 - \xi_2 \\ c_8 &= c_4^2 + \xi_3 \\ c_9 &= c_3 c_7 + c_3^2 c_8 + c_6 \\ c_{10} &= c_1 + 2c_4 + 2\sqrt{c_8} \\ c_{11} &= c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8}) \\ c_{12} &= c_4 + \sqrt{c_8} \\ c_{13} &= c_5 - (\sqrt{c_9} + c_3\sqrt{c_8}) \end{aligned} \quad (6)$$

(ii) Essential polynomial functions

$$\pi(s) = c_4 + c_5 s - \left[(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8} \right] \quad (7)$$

$$k = -(c_7 + 2c_3 c_8) - 2\sqrt{c_8 c_9} \quad (8)$$

$$\begin{aligned} \tau(s) &= c_1 + 2c_4 - (c_2 - 2c_5)s \\ &\quad - 2[(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}] \end{aligned} \quad (9)$$

$$\tau'(s) = -2c_3 - 2(\sqrt{c_9} + c_3\sqrt{c_8}) < 0 \quad (10)$$

(iii) Energy equation

$$\begin{aligned} c_2 n - (2n+1)c_5 + 2n+1(\sqrt{c_9} + c_3\sqrt{c_8}) \\ + n(n-1)c_3 + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0 \end{aligned} \quad (11)$$

(iv) Wave functions

$$\rho(s) = s^{c_{10}}(1 - c_3 s)^{c_{11}} \quad (12)$$

$$\phi(s) = s^{c_{12}}(1 - c_3 s)^{c_{13}}, c_{12} > 0, c_{13} > 0 \quad (13)$$

$$y_n(s) = P_n^{(c_{10}, c_{11})}(1 - 2c_3 s), c_{10} > -1, c_{11} > -1 \quad (14)$$

$$\psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}} (1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} P_n^{(c_{10}-1, \frac{c_{11}}{c_3} - c_{10}-1)}(1 - 2c_3 s) \quad (15)$$

Where, $P_n^{(\mu, \nu)}(x)$, $\mu > -1$, $\nu > -1$, and $x \in [-1, 1]$ are Jacobi polynomials with

$$P_n^{(\alpha, \beta)}(1 - 2s) = \frac{(\alpha+1)_n}{n!} {}_2F_1(-n, 1 + \alpha + \beta + n; \alpha + 1; s) \quad (16)$$

Here, $N_{n\kappa}$ are the normalization constants. Also, the above wave functions can be expressed in terms of the hyper-geometric function via

$$\psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}} (1 - c_3 s)^{c_{13}} {}_2F_1(-n, 1 + c_{10} + c_{11} + n; c_{10} + 1; c_3 s) \quad (17)$$

Where, $c_{12} > 0, c_{13} > 0$ and $s \in [0, 1/c_3], c_3 \neq 0$.

2.2. Theory of Dirac Equation

The Dirac equation for spin- $\frac{1}{2}$ particles moving in an attractive scalar potential $S(r)$, a repulsive vector potential $V(r)$ and a tensor potential $U(r)$ in the relativistic unit ($\hbar = c = 1$) is [17-19] is

$$[\vec{\alpha} \cdot \vec{p} + \beta(M + S(r)) - i\beta\vec{\alpha}\vec{r}U(r)]\psi(r) = [E - V(r)]\psi(r) \quad (18)$$

Where, E is the relativistic energy of the system, $\vec{p} = -i\vec{\nabla}$ is the three dimensional momentum operator, M is the mass of the Fermi particle, and $\vec{\alpha}, \beta$ are the 4×4 Dirac matrices given as

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma}_i \\ \vec{\sigma}_i & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (19)$$

Where, I is a 2×2 unitary matrix and $\vec{\sigma}_i$ are the Pauli three-vector matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (20)$$

The eigen-values of the spin-orbit coupling operator are $\kappa = \left(j + \frac{1}{2}\right) > 0, \kappa = -\left(j + \frac{1}{2}\right) < 0$ for unaligned $j = l - \frac{1}{2}$ and the aligned spin $j = l + \frac{1}{2}$, respectively. The set (H, K, J^2, J_z) forms a complete set of conserved quantities. Thus, we can write the spinors [19] as

$$\psi_{n\kappa}(r) = \frac{1}{r} \begin{pmatrix} F_{n\kappa}(r) Y_{jm}^l(\theta, \varphi) \\ iG_{n\kappa}(r) Y_{jm}^{\bar{l}}(\theta, \varphi) \end{pmatrix} \quad (21)$$

Where, $F_{n\kappa}(r), G_{n\kappa}(r)$ represent the upper and lower components of the Dirac spinors. $Y_{jm}^l(\theta, \varphi), Y_{jm}^{\bar{l}}(\theta, \varphi)$ are the spin and pseudo-spin spherical harmonics and m is the projection on the z -axis. Using well-known identities [18-19],

$$\begin{aligned} (\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) &= \vec{A} \cdot \vec{B} + i \vec{\sigma} \cdot (\vec{A} \times \vec{B}), \\ \vec{\sigma} \cdot \vec{p} &= \vec{\sigma} \cdot \hat{r} \left(\hat{r} \cdot \vec{p} + i \frac{\vec{\sigma} \cdot \vec{L}}{r} \right) \end{aligned} \quad (22)$$

as well as the following relations

$$\begin{aligned} (\vec{\sigma} \cdot \vec{L}) Y_{jm}^{\bar{l}}(\theta, \varphi) &= (\kappa - 1) Y_{jm}^{\bar{l}}(\theta, \varphi) \\ (\vec{\sigma} \cdot \vec{L}) Y_{jm}^l(\theta, \varphi) &= -(\kappa - 1) Y_{jm}^l(\theta, \varphi) \\ (\vec{\sigma} \cdot \hat{r}) Y_{jm}^l(\theta, \varphi) &= -Y_{jm}^{\bar{l}}(\theta, \varphi) \\ (\vec{\sigma} \cdot \hat{r}) Y_{jm}^{\bar{l}}(\theta, \varphi) &= -Y_{jm}^l(\theta, \varphi) \end{aligned} \quad (23)$$

we find the following two coupled first-order Dirac equation [18-19]

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{n\kappa}(r) = (M + E_{n\kappa} - \Delta(r)) G_{n\kappa}(r) \quad (24)$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{n\kappa}(r) = (M - E_{n\kappa} + \Sigma(r)) F_{n\kappa}(r) \quad (25)$$

Where,

$$\Delta(r) = V(r) - S(r) \quad (26)$$

$$\Sigma(r) = V(r) + S(r) \quad (27)$$

Eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ in Eqns. (24) and (25), we obtain the second-order a Schrödinger-like equation

$$\left\{ \begin{aligned} & \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa U(r)}{r} - \frac{dU(r)}{dr} - U^2(r) - (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) \\ & + \frac{\frac{d\Delta(r)}{dr} \left(\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right)}{(M + E_{n\kappa} - \Delta(r))} \end{aligned} \right\} F_{n\kappa}(r) = 0 \quad (28)$$

$$\left\{ \begin{aligned} & \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa U(r)}{r} + \frac{dU(r)}{dr} - U^2(r) - (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) \\ & + \frac{\frac{d\Sigma(r)}{dr} \left(\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right)}{(M - E_{n\kappa} + \Sigma(r))} \end{aligned} \right\} G_{n\kappa}(r) = 0 \quad (29)$$

Where, $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$, $\kappa(\kappa+1) = l(l+1)$.

3. Results and Discussion

3.1. Pseudo-spin and spin symmetry limits under Coulomb-like tensor interaction

In this section, we intend to investigate the Dirac equation with modified Tietz potential in the presence of Coulomb tensor interactions.

3.2. Pseudo-spin symmetry in the Dirac equation with Coulomb-like tensor interaction

The pseudo-spin symmetry occurs in the Dirac equation when $\frac{d\Sigma(r)}{dr} = 0$, or equivalently, $\Sigma(r) = C_{ps} = \text{const}$ [1-4]. To investigate the approximate analytical solution of the Tietz potentials, we consider the sum of the scalar and vector potential [20,25] as

$$\Delta(r) = D_e + D_e \frac{(a+b)e^{-4\alpha r} - be^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \quad (30)$$

This is in addition to the Coulomb-tensor interaction term [17]

$$U(r) = -\frac{H}{r}, r \geq R_e \quad (31)$$

With

$$H = \frac{z_a z_b e^2}{4\pi\epsilon_0} \quad (32)$$

Where, R_e is the Coulomb radius, z_a and z_b denote the charges of the projectile a and the target nuclei b , respectively [16-17].

Substituting the above equations into Eqn. (29) yields

$$\left\{ \frac{d^2}{dr^2} - \frac{\Lambda_\kappa(\Lambda_\kappa-1)}{r^2} - (M + E_{n,\kappa}^{ps})(M - E_{n,\kappa}^{ps} + C_{ps}) \right\} G_{n,\kappa}^{ps}(r) + (M - E_{n,\kappa}^{ps} + C_{ps}) \left[D_e + D_e \frac{(a+b)e^{-4\alpha r} - be^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \right] G_{n,\kappa}^{ps}(r) = 0. \quad (33)$$

Where, $\Lambda_\kappa = (\kappa + H)$.

It is well known that the above equation cannot be solved exactly due to the centrifugal term, r^{-2} . In order to get rid of the centrifugal term, we make use of the following approximation [30]

$$\frac{1}{r^2} \approx \frac{4\alpha^2 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \quad (34)$$

Substituting Eqn. (34) into Eqn. (33) in view of the transformation, $s = e^{-2\alpha r}$, yields

$$\frac{d^2 G_{n,\kappa}^{ps}}{dr^2} + \frac{(1-s)}{s(1-s)} \frac{dG_{n,\kappa}^{ps}}{dr} + \frac{1}{s^2(1-s)^2} [-\rho_1^{ps}s^2 + \rho_2^{ps}s - \rho_3^{ps}] G_{n,\kappa}^{ps} = 0 \quad (35)$$

Where,

$$\rho_1^{ps} = -\frac{1}{4\alpha^2}(-\varepsilon_{ps}^2 + \beta_{ps}D_e(1+a+b)) \quad (36)$$

$$\rho_2^{ps} = -\Lambda_\kappa(\Lambda_\kappa - 1) + \frac{1}{4\alpha^2}(2\varepsilon_{ps}^2 - \beta_{ps}D_e(2+b)) \quad (37)$$

$$\rho_3^{ps} = -\frac{1}{4\alpha^2}(-\varepsilon_{ps}^2 + \beta_{ps}D_e) \quad (38)$$

and

$$\beta_{ps} = (M - E_{n,\kappa}^{ps} + C_{ps}), \varepsilon_{ps}^2 = (M + E_{n,\kappa}^{ps})\beta_{ps}$$

$$\left\{ \frac{d^2}{dr^2} - \frac{\eta_\kappa(\eta_\kappa - 1)}{r^2} - \varepsilon^2 - \beta \left[D_e + D_e \frac{(a+b)e^{-4\alpha r} - be^{-2\alpha r}}{(1 - e^{-2\alpha r})^2} \right] \right\} F_{n,\kappa}^s(r) = 0 \quad (40)$$

Where,

$$\begin{aligned} \eta_\kappa &= (\kappa + H + 1) \\ \varepsilon_s^2 &= (M - E_{n,\kappa}^s)(M + E_{n,\kappa}^s - C_s) \\ \beta_s &= (M + E_{n,\kappa}^s)C_s \end{aligned} \quad (41)$$

By using the approximation of Eqn. (34) for the centrifugal term in Eqn. (40), then Eqn. (40) leads to the following second order differential equation in view of the $z = e^{-2\alpha r}$ transformation

$$\begin{aligned} \frac{d^2 F_{n,\kappa}^s}{dz^2} + \frac{(1-z)}{z(1-z)} \frac{dF_{n,\kappa}^s}{dz} \\ + \frac{1}{z^2(1-z)^2} [-\delta_1^s z^2 + \delta_2^s z - \delta_3^s] F_{n,\kappa}^s(z) = 0 \end{aligned} \quad (42)$$

Where,

$$\delta_1^s = \left(\frac{\varepsilon^2}{4\alpha^2} + \frac{\beta D_e}{4\alpha^2} + \frac{\beta D_e(a+b)}{4\alpha^2} \right) \quad (43)$$

$$\delta_2^s = \left(\frac{\varepsilon^2}{2\alpha^2} + \frac{\beta D_e}{2\alpha^2} + \frac{\beta D_e b}{4\alpha^2} - \eta_\kappa(\eta_\kappa - 1) \right) \quad (44)$$

$$\delta_3^s = \left(\frac{\varepsilon^2}{4\alpha^2} + \frac{\beta D_e}{4\alpha^2} \right) \quad (45)$$

3.3. Spin symmetry in the Dirac equation with Coulomb-like tensor interaction

In the case of spin symmetry limit, we use the following scalar, vector and tensor potentials

$$\begin{aligned} \Sigma(r) &= D_e + D_e \frac{(a+b)e^{-4\alpha r} - be^{-2\alpha r}}{(1 - e^{-2\alpha r})^2}, \Delta(r) = C \\ U(r) &= -\frac{H}{r} \end{aligned} \quad (39)$$

Substituting Eqn. (39) into Eqn. (28) yields,

3.4. Pseudo-spin and spin symmetry solutions with Coulomb interaction

We shall now solve Eqns. (35) and (42) by using the parametric generalization of the NU method [20] in the sequel.

3.4.1. Pseudo-spin symmetry solution with Coulomb interaction

Now, comparing Eqn. (3) with Eqn. (35), we obtain

$$\begin{aligned} c_1 = 1, c_2 = 1, c_3 = 1, \\ \xi_1 = \rho_1^{ps}, \xi_2 = \rho_2^{ps}, \xi_3 = \rho_3^{ps} \end{aligned} \quad (46)$$

Other parameters can be obtain from Eqn. (6) as

$$\begin{aligned} c_4 = 0, c_5 = -\frac{1}{2}, c_6 = \frac{1}{4} + \rho_1^{ps}, c_7 = -\rho_2^{ps}, \\ c_8 = \rho_3^{ps}, c_9 = \frac{1}{4} + \rho_1^{ps} + \rho_3^{ps} - \rho_2^{ps}, c_{10} = 1 + 2\sqrt{\rho_3^{ps}}, \\ c_{11} = 2 + 2 \left[\sqrt{\frac{1}{4} + \rho_1^{ps} + \rho_3^{ps} - \rho_2^{ps}} + \sqrt{\rho_3^{ps}} \right], c_{12} = \sqrt{\rho_3^{ps}}, \\ c_{13} = -\frac{1}{2} - \left(\sqrt{\frac{1}{4} + \rho_1^{ps} + \rho_3^{ps} - \rho_2^{ps}} + \sqrt{\rho_3^{ps}} \right) \end{aligned} \quad (47)$$

Substituting Eqns. (46) and (47) into Eqn. (11) yields

$$n_2 + \left(n + \frac{1}{2}\right)(2n+1) \left[\sqrt{\frac{1}{4} + \rho_1^{ps} + \rho_3^{ps} - \rho_2^{ps} + \rho_3^{ps}} \right] - \rho_2^{ps} + 2\rho_3^{ps} + 2\sqrt{\rho_3^{ps} \left(\frac{1}{4} + \rho_1^{ps} + \rho_3^{ps} - \rho_2^{ps} \right)} = 0 \quad (48)$$

From Eqns. (15) and (16) the lower component of the wave functions are as

$$G_{n,\kappa}^{ps}(r) = N_{n,\kappa}^{ps} e^{-2\alpha\sqrt{\rho_3^{ps}}r} \left(1 - e^{-2\alpha r}\right)^{\frac{1}{2} + \sqrt{\frac{1}{4} + \rho_1^{ps} + \rho_3^{ps} - \rho_2^{ps}}} P_n^{(2\sqrt{\rho_3^{ps}}, 2\sqrt{\frac{1}{4} + \rho_1^{ps} + \rho_3^{ps} - \rho_2^{ps}})} (1 - 2e^{-2\alpha r}) \quad (49)$$

The other component of the wave function can be obtained as

$$F_{n,\kappa}^{ps}(r) = \frac{1}{M - E_{n,\kappa}^{ps} + C_{ps}} \left(\frac{d}{dr} - \frac{\kappa}{r} - \frac{H}{r} \right) G_{n,\kappa}^{ps}(r) \quad (50)$$

Where, $N_{n,\kappa}^{ps}$ is the normalization constant and $E_{n,\kappa}^{ps} \neq M + C_{ps}$

3.4.2. Spin symmetry solution with Coulomb interaction

We applied the same procedure that was used for Eqn. (35) to Eqn. (42) for the spin symmetry limits in order to avoid repetition. The energy eigenvalues equation and the corresponding upper wave function of the Dirac theory for the Tietz potential in the presence of Coulomb tensor interaction are obtained as

$$n^2 + \left(n + \frac{1}{2}\right)(2n+1) \left[\sqrt{\frac{1}{4} + \delta_1^s + \delta_3^s - \delta_2^s + \sqrt{\delta_3^s}} \right] - \delta_2^s + 2\delta_3^s + 2\sqrt{\delta_3^s \left(\frac{1}{4} + \delta_1^s + \delta_3^s - \delta_2^s \right)} = 0 \quad (51)$$

$$F_{n,\kappa}^s(r) = N_{n,\kappa}^s e^{-2\alpha\sqrt{\delta_3^s}r} \left(1 - e^{-2\alpha r}\right)^{\frac{1}{2} + \sqrt{\frac{1}{4} + \delta_1^s + \delta_3^s - \delta_2^s}} P_n^{(2\sqrt{\delta_3^s}, 2\sqrt{\frac{1}{4} + \delta_1^s + \delta_3^s - \delta_2^s})} (1 - 2e^{-2\alpha r}) \quad (52)$$

Where, $N_{n,\kappa}^s$ is the normalization constant.

The other component of the Dirac spinor can be found as

$$G_{n,\kappa}^s(r) = \frac{1}{M + E_{n,\kappa}^s - C_s} \left(\frac{d}{dr} + \frac{\kappa}{r} + \frac{H}{r} \right) F_{n,\kappa}^s(r) \quad (53)$$

In Tables 1 and 2, we have reported the energy of the Dirac equation for pseudo-spin and spin symmetries in the presence and absence of Coulomb-like tensor interaction for Tietz potential. In the absence of this tensor interaction, there are degeneracies between $(1s_{1/2}, 0d_{3/2})$, $(1p_{3/2}, 0f_{5/2})$... in pseudo-spin symmetry and $(0p_{1/2}, 0p_{3/2})$, $(0d_{3/2}, 0d_{5/2})$... in spin symmetry limit. When the Coulomb-like tensor interaction appears these degeneracies disappear. We have shown the effect of tensor interaction on various components of the Dirac spinor in both pseudo-spin and spin symmetries in Figs. 1 and 2. We have also indicated the behavior of the energy of the system versus H

for spin and pseudo-spin symmetries in Fig. 5. It is clear that when $H=0$, we have degeneracies between $(1p_{3/2}, 0f_{5/2})$, $(1d_{5/2}, 0g_{7/2})$, $(0d_{3/2}, 0d_{5/2})$ and $(0f_{7/2}, 0f_{5/2})$, and when H increases these degeneracies disappear.

3.5. Pseudo-spin and spin symmetry limits under Yukawa-like tensor interaction

In the following section, we intend to investigate the Dirac equation with Tietz potential in the presence of Yukawa-like tensor interactions. The presence of Yukawa potential as the tensor term in the Dirac equation also removes degeneracies in addition to the Coulomb tensor interaction. Thus, it is pertinent to investigate the Tietz potential with Yukawa potential as tensor interaction term.

Table 1: The energy of the pseudo-spin symmetry limit in the presence and absence of Coulomb-like tensor interaction for $M = 5(\text{fm}^{-1})$, $\alpha = 0.1(\text{fm}^{-1})$, $C_{ps} = -5(\text{fm}^{-1})$, $D_e = 0.5$, $\alpha = -5$, $b = -3$.

$\tilde{l}$	n, k	state	$E_{n,k}^{ps}(H=0)$	$E_{n,k}^{ps}(H=0.5)$	$n-1, k$	state	$E_{n-1,k}^{ps}(H=0)$	$E_{n-1,k}^{ps}(H=0.5)$
1	1, -1	$1s_{\frac{1}{2}}$	-4.4155 09205	-4.413092664 -0.000021707	0, 2	$0d_{\frac{3}{2}}$	-4.415509205	-4.418813686 -0.000011250
2	1, -2	$1p_{\frac{3}{2}}$	-4.422930447 -0.000135959	-4.418813686 -0.000011250	0, 3	$0f_{\frac{5}{2}}$	-4.422930447 -0.000135959	-4.427766271 -0.000536239
3	1, -3	$1d_{\frac{5}{2}}$	-4.433213042 -0.001354860	-4.427766271 -0.000536239	0, 4	$0g_{\frac{7}{2}}$	-4.433213042 -0.001354860	-4.439150618 -0.002701492
4	1, -4	$1f_{\frac{7}{2}}$	-4.445449751 -0.004657738	-4.439150618 -0.002701492	0, 5	$0h_{\frac{9}{2}}$	-4.445449751 -0.004657738	-4.451974929 -0.007284436

Table 2: The energy of the spin symmetry limit in the presence and absence of Coulomb-like tensor interaction for $M = 5(\text{fm}^{-1})$, $\alpha = 0.1(\text{fm}^{-1})$, $C_s = 5(\text{fm}^{-1})$, $D_e = 0.5$, $\alpha = -5$, $b = -3$.

l	n, k	state	$E_{n,k}^s(H=0)$	$E_{n,k}^s(H=0.5)$	n, k	state	$E_{n,k}^s(H=0)$	$E_{n,k}^s(H=0.5)$
1	0, -2	$0p_{\frac{3}{2}}$	0.0345246396 0.0004856573	0.0158594389 0.0000520760	0, 1	$0p_{\frac{1}{2}}$	0.0345246396 0.0004856573	0.0592645907 0.0016212192
2	0, -3	$0d_{\frac{5}{2}}$	0.0897374758 0.0036479590	0.0592645907 0.0016212192	0, 2	$0d_{\frac{3}{2}}$	0.0897374758 0.0036479590	0.1257210612 0.0066827770
3	0, -4	$0f_{\frac{7}{2}}$	0.1670333382 0.0108044271	0.1257210612 0.0066827770	0, 3	$0f_{\frac{5}{2}}$	0.1670333382 0.0108044271	0.2135013911 0.0160705373
4	0, -5	$0g_{\frac{9}{2}}$	0.2649462667 0.0225266924	0.2135013911 0.0160705373	0, 4	$0g_{\frac{7}{2}}$	0.2649462667 0.0225266924	0.3211741576 0.0302116722

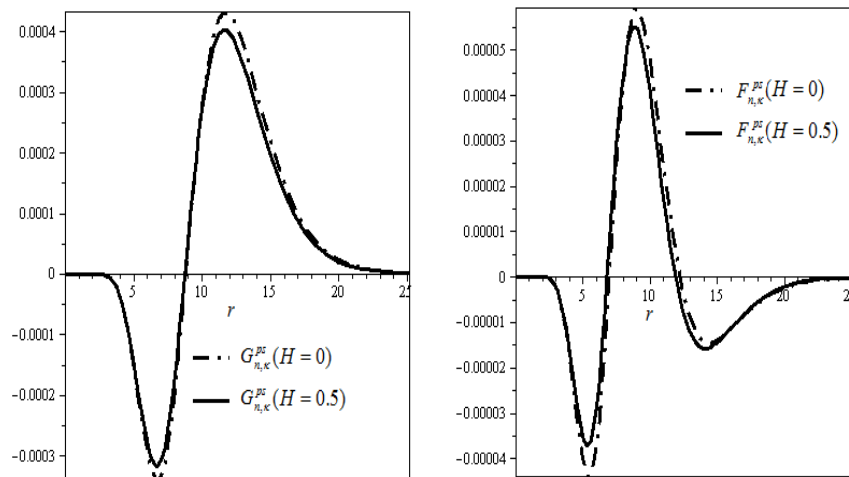


Fig.1: Wave functions of $1s_{\frac{1}{2}}$ in the pseudo-spin symmetry in the presence and absence of Coulomb tensor interaction for $M = 5(\text{fm}^{-1})$, $\alpha = 0.1(\text{fm}^{-1})$, $C_{ps} = -5(\text{fm}^{-1})$, $D_e = 0.5$, $a = -5$, $b = -3$.

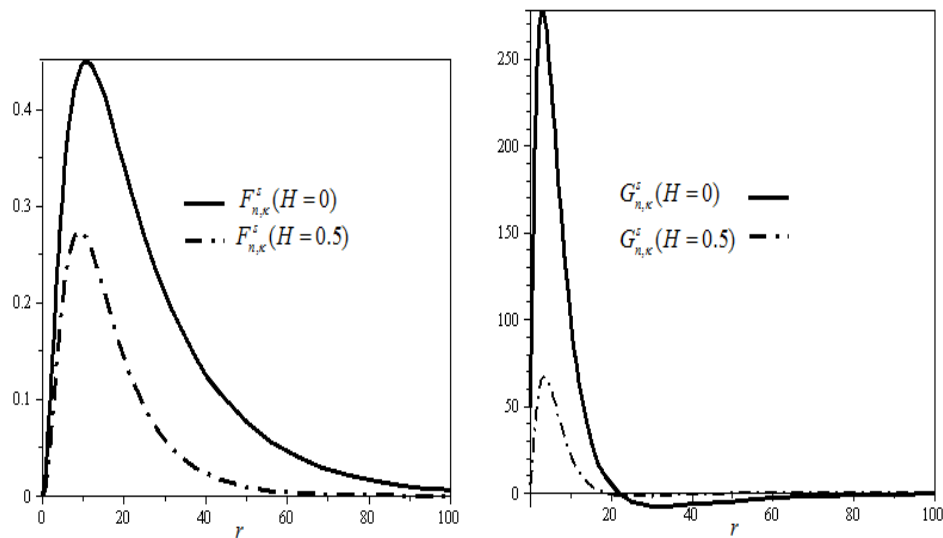


Fig.2: Wave functions of $0p_{\frac{1}{2}}$ in the spin symmetry in the presence and absence of Coulomb-like tensor interaction for

$$M = 5(\text{fm}^{-1}), \alpha = 0.1(\text{fm}^{-1}), C_s = 5(\text{fm}^{-1}), D_e = 0.5, a = -5, b = -3.$$

3.5.1. Pseudo-spin symmetry in the Dirac equation with Yukawa tensor interaction

The pseudo-spin symmetry occurs in the Dirac theory as $\frac{d\Sigma(r)}{dr} = 0$ or equivalently

$\Sigma(r) = C_{ps} = \text{const}$ [1-4]. In order to find the approximate analytical solution of the Dirac equation under the pseudo-spin symmetry limit, we take the difference of the scalar and vector potentials as the Tietz potentials

$$\Delta(r) = D_e + D_e \frac{(a+b)e^{-4\alpha r} - be^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \quad (54)$$

This is in addition to the Yukawa tensor interaction[31]

$$U(r) = -\frac{V_0 e^{-\alpha r}}{r} \quad (55)$$

Inserting Eqns. (54) and (55) into Eqn. (29) yields

$$\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \frac{2\kappa V_0 e^{-\alpha r}}{r^2} + \frac{\alpha V_0 e^{-\alpha r}}{r} \right.$$

$$\left. + \frac{V_0 e^{-\alpha r}}{r^2} - \frac{V_0^2 e^{-\alpha r}}{r^2} \right\} G_{n,\kappa}^{ps}(r) + \left(-\varepsilon_{ps}^2 + \beta \left[D_e + D_e \frac{(a+b)e^{-4\alpha r} - be^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \right] \right) G_{n,\kappa}^{ps}(r) = 0 \quad (56)$$

Where, $\varepsilon_{ps}^2 = (M + E_{n,\kappa}^{ps})(M - E_{n,\kappa}^{ps} + C_{ps})$ and $\beta_{ps} = (M - E_{n,\kappa}^{ps} + C_{ps})$.

Again, Eqn. (56) cannot be solved exactly by any known method because of the centrifugal term, $\frac{1}{r^2}$.

In order to get the approximate solution of Eqn. (56), we use the Pekeris approximation [32] given as

$$\frac{1}{r^2} \approx \frac{4\alpha^2 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \quad (57)$$

$$\frac{1}{r^2} \approx \frac{4\alpha^2 e^{-\alpha r}}{(1-e^{-2\alpha r})^2} \quad (58)$$

Substituting Eqns. (57) and (58) into Eqn. (56) and applying the transformation, $s = e^{-2\alpha r}$, we get

$$\frac{dG_{n,\kappa}^{ps}}{ds^2} + \frac{(1-s)}{s(1-s)} \frac{dG_{n,\kappa}^{ps}}{ds} + \frac{1}{s^2(1-s)^2} [-\gamma_1^{ps}s^2 + \gamma_2^{ps}s - \gamma_3^{ps}] G_{n,\kappa}^{ps}(s) = 0 \quad (59)$$

Where

$$\gamma_1^{ps} = \left(\frac{\epsilon_s^2}{4\alpha^2} - \frac{\beta_s D_e (a+b)}{4\alpha^2} - \frac{\beta D_e}{4\alpha^2} + V_0 \left(V_0 + \frac{1}{2} \right) \right) \quad (60)$$

$$\gamma_2^{ps} = - \left(\kappa(\kappa-1) + \left(2\kappa - \frac{3}{2} \right) V_0 + \frac{\beta_s D_e}{2\alpha^2} + \frac{\beta_s D_e b}{4\alpha^2} - \frac{\epsilon_s^2}{2\alpha^2} \right) \quad (61)$$

$$\gamma_3^{ps} = - \left(\frac{\beta_s D_e}{4\alpha^2} - \frac{\epsilon_s^2}{4\alpha^2} \right) \quad (62)$$

3.5.2. Spin symmetry in the Dirac equation with Yukawa potential

In the spin symmetry limit case, we use the following scalar, vector and tensor potentials given as

$$\Delta(r) = C_s, \Sigma(r) = D_e + D_e \frac{(a+b)e^{-4\alpha r} - be^{-2\alpha r}}{(1-e^{-2\alpha r})^2}$$

$$U(r) = - \frac{V_0 e^{-\alpha r}}{r} \quad (63)$$

Which, transform Eqn. (27) into the form

$$\begin{aligned} \frac{d_2 F_{n,\kappa}^s}{dr^2} + \frac{(1-s)}{s(1-s)} \frac{dF_{n,\kappa}^s}{dr} \\ + \frac{1}{s^2(1-s)^2} [-\sigma_1^s s^2 + \sigma_2^s s - \sigma_3^s] F_{n,\kappa}^s(s) = 0 \end{aligned} \quad (64)$$

Where,

$$\sigma_1^s = \left(\frac{\epsilon_s^2}{4\alpha^2} + \frac{\beta_s D_e}{4\alpha^2} + \frac{\beta_s D_e (a+b)}{4\alpha^2} + V_0 \left(V_0 - \frac{1}{2} \right) \right) \quad (65)$$

$$\sigma_2^s = \left(\frac{2\epsilon_s^2}{4\alpha^2} + \frac{2\beta_s D_e}{4\alpha^2} + \frac{\beta_s D_e b}{4\alpha^2} - \kappa(\kappa+1) - \left(2\kappa + \frac{3}{2} \right) V_0 \right) \quad (66)$$

$$\sigma_3^s = \left(\frac{\epsilon_s^2}{4\alpha^2} + \frac{\beta_s D_e}{4\alpha^2} \right) \quad (67)$$

3.6. Pseudo-spin and spin symmetry solutions

In this section, we intend to investigate the solutions of Eqns. (59) and (64) for pseudo-spin and spin symmetry using the parametric generalization of the NU method.

3.6.1. Pseudo-spin symmetry solution

Comparing Eqns. (59) and (3), we obtain

$$c_1 = 1, c_2 = 1, c_3 = 1 \quad (68)$$

$$\xi_1 = \gamma_1^{ps}, \xi_2 = \gamma_2^{ps}, \xi_3 = \gamma_3^{ps}$$

And, other parameters are obtained as

$$c_4 = 0, c_5 = -\frac{1}{2}, c_6 = \frac{1}{4} + \gamma_1^{ps}, c_7 = -\gamma_2^{ps}, c_8 = \gamma_3^{ps},$$

$$c_9 = \frac{1}{4} + \gamma_1^{ps} + \gamma_3^{ps} - \gamma_2^{ps}, c_{10} = 1 + 2\sqrt{\gamma_3^{ps}},$$

$$c_{11} = 2 + 2 \left[\sqrt{\frac{1}{4} + \gamma_1^{ps} + \gamma_3^{ps} - \gamma_2^{ps}} + \sqrt{\gamma_3^{ps}} \right], c_{12} = \sqrt{\gamma_3^{ps}},$$

$$c_{13} = -\frac{1}{2} - \left[\sqrt{\frac{1}{4} + \gamma_1^{ps} + \gamma_3^{ps} - \gamma_2^{ps}} + \sqrt{\gamma_3^{ps}} \right] \quad (69)$$

Substituting Eqns. (68) and (69) in Eqn. (11) gives the energy eigen-values as

$$\begin{aligned} n(n+1) + \frac{1}{2} + (2n+1) \left(\sqrt{\frac{1}{4} + \gamma_1^{ps} + \gamma_3^{ps} - \gamma_2^{ps}} + \sqrt{\gamma_3^{ps}} \right) \\ - \gamma_2^{ps} + 2\gamma_3^{ps} + 2\sqrt{-\gamma_3^{ps}\gamma_2^{ps} + (\gamma_3^{ps})^2} + \gamma_1^{ps}\gamma_3^{ps} + \frac{\gamma_3^{ps}}{4} = 0 \end{aligned} \quad (70)$$

From Eqn. (17), lower and upper components of the wave function are obtained as

$$G_{n,\kappa}^{ps'}(r) = N_{n,\kappa}^{ps} e^{-2\alpha\sqrt{\gamma_3^{ps}}r} (1-e^{-2\alpha r})^{\frac{1}{2} + \sqrt{\frac{1}{4} + \gamma_1^{ps} + \gamma_3^{ps} - \gamma_2^{ps}}} P_n \left(2\sqrt{\gamma_3^{ps}}, 2\sqrt{\frac{1}{4} + \gamma_1^{ps} + \gamma_3^{ps} - \gamma_2^{ps}} \right) (1-2e^{-2\alpha r}) \quad (71)$$

$$F_{n,\kappa}^{ps}(r) = \frac{1}{M - E_{n,\kappa}^{ps} + C_{ps}} \left(\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{n,\kappa}^{ps}(r) \quad (72)$$

Where, $E_{n,\kappa}^{ps} \neq M + C_{ps}$.

3.6.2. Spin symmetry solution

Also comparing Eqns. (64) and (3), we obtained

$$c_1 = 1, c_2 = 1, c_3 = 1 \quad (73)$$

$$\xi_1 = \sigma_1^s, \xi_2 = \sigma_2^s, \xi_3 = \sigma_3^s$$

From Eqn. (6) other parameters are obtained as

$$c_4 = 0, c_5 = -\frac{1}{2}, c_6 = \frac{1}{4} + \sigma_1^s, c_7 = -\sigma_2^s, c_8 = \sigma_3^s,$$

$$c_9 = \frac{1}{4} + \sigma_1^s + \sigma_3^s - \sigma_2^s, c_{10} = 1 + 2\sqrt{\sigma_3^s},$$

$$F_{n,\kappa}^s(r) = N_{n,\kappa}^s e^{-2\alpha\sqrt{\sigma_3^s}r} (1 - e^{-2\alpha r})^{\frac{1}{2} + \sqrt{\frac{1}{4} + \sigma_1^s + \sigma_3^s - \sigma_2^s}} P_n^{\left(2\sqrt{\sigma_3^s}, 2\sqrt{\frac{1}{4} + \sigma_1^s + \sigma_3^s - \sigma_2^s}\right)} (1 - 2e^{-2\alpha r}) \quad (77)$$

$$G_{n,\kappa}^s(r) = \frac{1}{M - E_{n,\kappa}^s - C_s} \left(\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{n,\kappa}^s(r) \quad (78)$$

The energy of the Dirac equation for the pseudo-spin and spin symmetries and Yukawa tensor interaction is given in Tables 3 and 4. It is obvious that we have degeneracies when $V_0 = 0$; and when $V_0 \neq 0$ these degeneracies disappear. We have also represented the effect of tensor interaction on the components of the Dirac spinor and the energy of the Dirac equation in both pseudo-spin and spin symmetries as shown in Figs. 3, 4 and 6.

3.7. Few special cases

In this section, we will study one special case of interest of the Tietz potential as follows. E. Maghsoodi et al. [33] have obtained the approximate solutions of the Dirac equation in the presence of Deng-Fan potential plus Coulomb tensor interaction term using the SUSYQM formalism. The general form of the Tietz potential in Eqn. (1) can be written as

$$c_{11} = 2 + 2 \left[\sqrt{\frac{1}{4} + \sigma_1^s + \sigma_3^s - \sigma_2^s} + \sqrt{\sigma_3^s} \right], c_{12} = \sqrt{\sigma_3^s},$$

$$c_{13} = -\frac{1}{2} - \left[\sqrt{\frac{1}{4} + \sigma_1^s + \sigma_3^s - \sigma_2^s} + \sqrt{\sigma_3^s} \right] \quad (75)$$

Substituting Eqns. (75) and (74) in Eqn.(11) gives the energy eigen-values as

$$n(n+1) + \frac{1}{2} + (2n+1) \left(\sqrt{\frac{1}{4} + \sigma_1^s + \sigma_3^s - \sigma_2^s} + \sqrt{\sigma_3^s} \right) - \sigma_2^s + 2\sigma_3^s + 2\sqrt{\sigma_3^s} \left(\frac{1}{4} + \sigma_1^s + \sigma_3^s - \sigma_2^s \right) = 0 \quad (76)$$

Finally, we obtain the upper and lower wave function as

$$V(r) = \frac{D_e - D_e(2+b)e^{-2\alpha r} + D_e(a+b+1)e^{-4\alpha r}}{(1 - e^{-2\alpha r})^2} \quad (79)$$

Comparing Eqn. (79) with the Deng-Fan potential [33],

$$V(r) = \frac{D_e(1 - 2(1+b)e^{-\alpha r} + D(b+1)^2 e^{-2\alpha r})}{(1 - e^{-\alpha r})^2} \quad (80)$$

We have: $D_e = D_e, (2+b)D_e = 2(1+b)D_e, (a+b+1)D_e = D_e(1+b)^2, 2\alpha \rightarrow \alpha$. When these parameters are substituted into the energy Eqns. (48) and (51) for pseudo-spin and spin symmetries, we obtained the energy values for the Deng-Fan potential. This result is consistent with the one obtained by Maghsoodi et al. [33].

4. Conclusions

In this work, we have obtained the approximate solutions of the Dirac equation for the Tietz potential within the Coulomb and Yukawa tensor interaction terms in the framework of pseudo-spin and spin symmetry limits using the parametric generalization of NU method. We have obtained the energies eigen-

values and the corresponding lower and upper wave functions expressed in terms of the hyper-geometric functions. Finally, the results of our work have been compared with the previous work of others authors in the literatures.

Competing interests

Authors declare that there is no competing conflict of interest.

Authors' contributions

ANI proposed the potential and wrote the manuscript with SEE. BHY and SZ carried out the numerical analysis while HH oversaw the write up and the numerical results. Finally, all the authors read and approved the final version of the manuscript.

Table 3: The energy of the pseudo-spin symmetry limit in the presence and absence of Yukawa tensor interaction for $M = 5(\text{fm}^{-1})$, $\alpha = 0.1(\text{fm}^{-1})$, $C_{ps} = -5(\text{fm}^{-1})$, $D_e = 0.5$, $a = -5$, $b = -3$.

$\tilde{l}$	n,k	state	$E_{n,k(V_0=0)}^{ps}$	$E_{n,k(V_0=0.6)}^{ps}$	$n-1,k$	state	$E_{n-1,k(V_0=0)}^{ps}$	$E_{n-1,k(V_0=0.6)}^{ps}$
1	1,-1	$1s_{\frac{1}{2}}$	-4.415509205	-4.411705907	0,2	$0d_{\frac{3}{2}}$	-4.415509205	-4.418608229
2	1,-2	$1p_{\frac{3}{2}}$	-4.422930447 -0.000135959	-4.417107867	0,3	$0f_{\frac{5}{2}}$	-4.422930447 -0.000135959	-4.427911344 -0.000368658
3	1,-3	$1d_{\frac{5}{2}}$	-4.433213042 -0.001354860	-4.425832200 -0.000189880	0,4	$0g_{\frac{7}{2}}$	-4.433213042 -0.001354860	-4.439575032 -0.002492867
4	1,-4	$1f_{\frac{7}{2}}$	-4.445449751 -0.004657738	-4.437100028 -0.001883434	0,5	$0h_{\frac{9}{2}}$	-4.445449751 -0.004657738	-4.452589478 -0.007142810

Table 4: The energy of the spin symmetry limit in the presence and absence of Yukawa tensor interaction for $M = 5(\text{fm}^{-1})$, $\alpha = 0.1(\text{fm}^{-1})$, $C_s = 5(\text{fm}^{-1})$, $D_e = 0.5$, $a = -5$, $b = -3$.

l	n,k	state	$E_{n,k(V_0=0)}^s$	$E_{n,k(V_0=0.6)}^s$	n,k	state	$E_{n,k(V_0=0)}^s$	$E_{n,k(V_0=0.6)}^s$
1	0,-2	$0p_{\frac{3}{2}}$	0.0345246396 0.0004856573	0.0129320817 0.0000094073	0,1	$0p_{\frac{1}{2}}$	0.0345246396 0.0004856573	0.0650771080 0.0018544678
2	0,-3	$0d_{\frac{5}{2}}$	0.0897374758 0.0036479590	0.0540002899 0.0012429871	0,2	$0d_{\frac{3}{2}}$	0.0897374758 0.0036479590	0.1338012089 0.0072727189
3	0,-4	$0f_{\frac{7}{2}}$	0.1670333382 0.0108044271	0.1183188257 0.0058543326	0,3	$0f_{\frac{5}{2}}$	0.1670333382 0.0108044271	0.2236884849 0.0170824745
4	0,-5	$0g_{\frac{9}{2}}$	0.2649462667 0.0225266924	0.2040845958 0.0147474678	0,4	$0g_{\frac{7}{2}}$	0.2649462667 0.0225266924	0.3333084913 0.0316862433

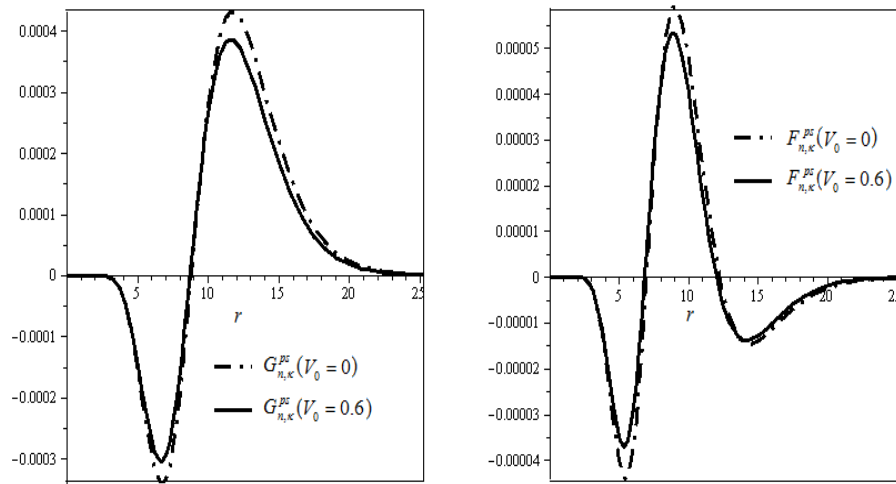


Fig.3: Wave functions of $1s_{\frac{1}{2}}$ in the pseudo-spin symmetry in the presence and absence of Yukawa tensor interaction for

$$M = 5(\text{fm}^{-1}), \alpha = 0.1(\text{fm}^{-1}), C_{ps} = -5(\text{fm}^{-1}), D_e = 0.5, a = -5, b = -3.$$

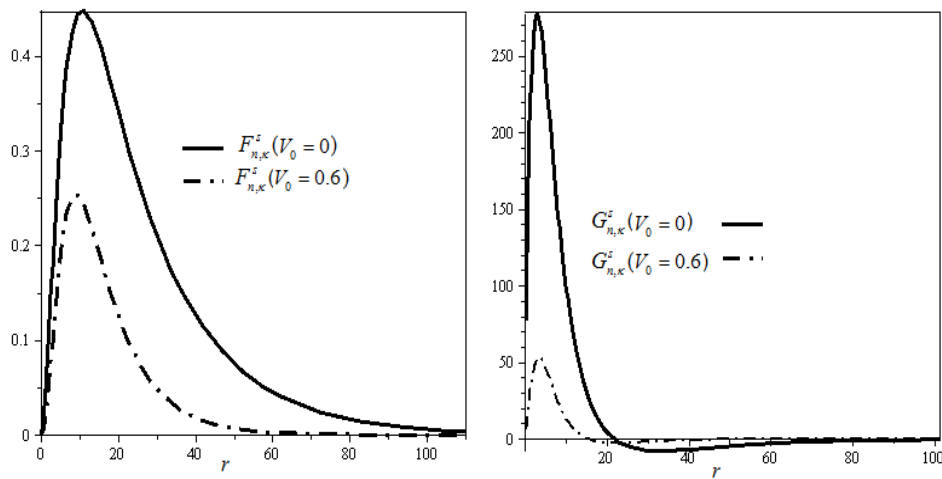


Fig.4: Wave functions of $0p_{\frac{1}{2}}$ in the spin symmetry in the presence and absence of Yukawa tensor interaction for

$$M = 5(\text{fm}^{-1}), \alpha = 0.1(\text{fm}^{-1}), C_s = 5(\text{fm}^{-1}), D_e = 0.5, a = -5, b = -3.$$

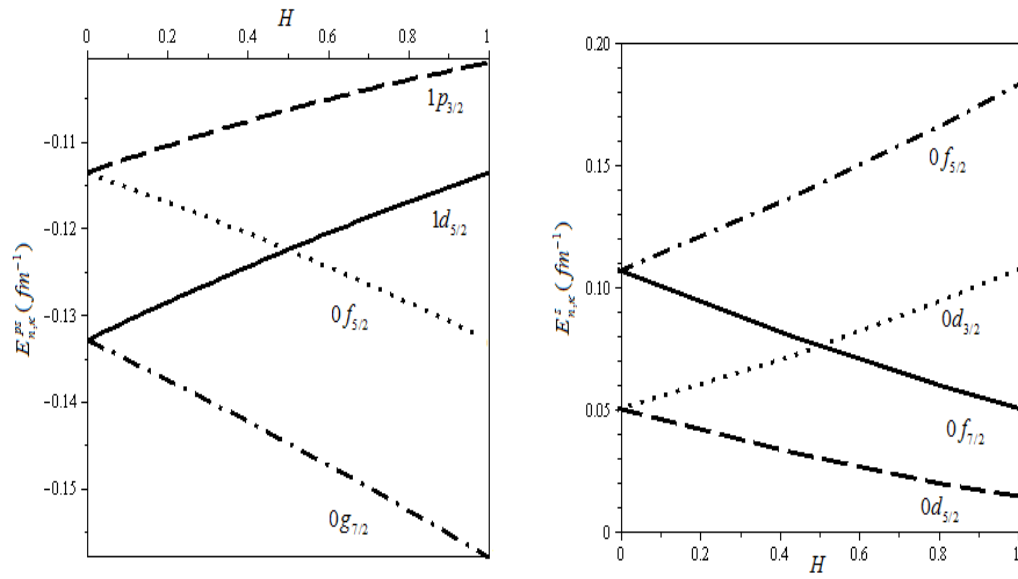


Fig.5: Energy spectra in the pseudo-spin and spin symmetry versus H for Coulomb-like tensor interaction for $M = 5(\text{fm}^{-1})$, $\alpha = 0.1(\text{fm}^{-1})$, $C_{ps} = -5(\text{fm}^{-1})$, $C_s = 5(\text{fm}^{-1})$, $D_e = 0.5$, $a = -5$, $b = -3$.

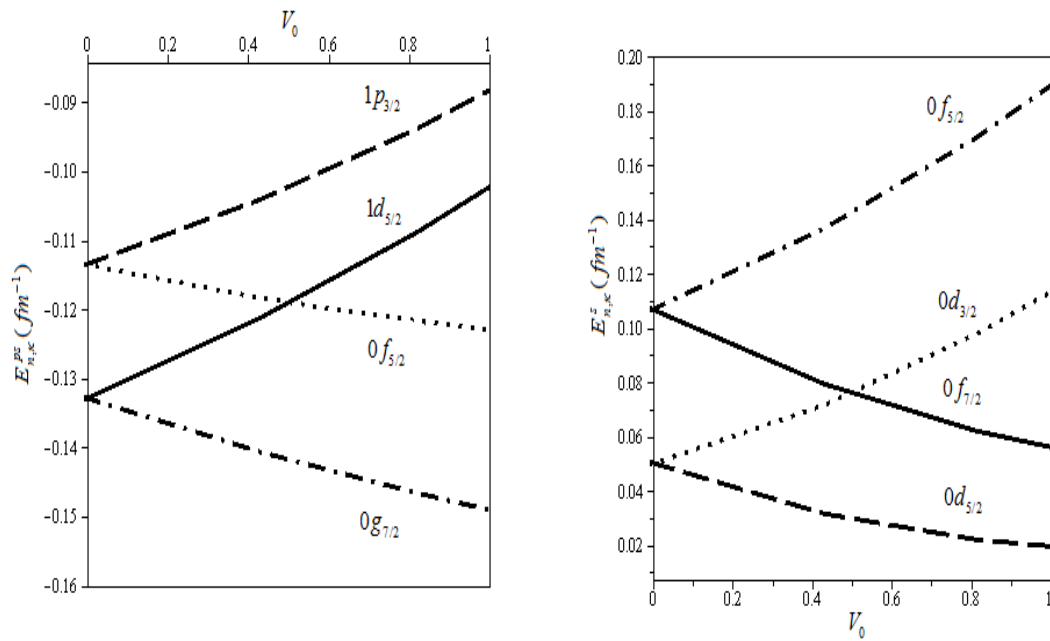


Fig.6: Energy spectra in the pseudo-spin and spin symmetry versus V_0 for Yukawa tensor interaction for $M = 5(\text{fm}^{-1})$, $\alpha = 0.1(\text{fm}^{-1})$, $C_{ps} = -5(\text{fm}^{-1})$, $C_s = 5(\text{fm}^{-1})$, $D_e = 0.5$, $a = -5$, $b = -3$.

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