

Numerical Solution of the Blasius Problem

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We present here some new aspects of the numerical solution of the Blasius problem using the standard Adomian Decomposition Method coupled to a Finite Difference Scheme. Each of the above mentioned methods, taken separately, is characterized by their respective advantages and disadvantages. Our main idea regarding this issue is applied to the Blasius equation. A discussion of the results obtained is presented.

1. The Blasius Problem: A Setup

The Blasius problem describes stationary and incompressible flow in two dimensions, forming the boundary layer on a semi-infinite flat plate parallel to the flow, which is a non-linear third order differential equation. The characteristic dimension in the direction parallel to flow is very large as compared to the characteristic dimension perpendicular to flow, the so-called boundary layer thickness.

Solving the Blasius equation has been studied extensively in [1], [2] and [3]. The present work goes beyond this classical objective and focuses on the exploration of a coupling between two numerical methods that is expected to be tested on particular cases before an eventual generalization [4].

For this purpose, we will briefly review the fundamentals governing this particular fluid flow. Thus, consider the following Navier Stokes differential equations [5]

$$\begin{cases} \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} = 0 \\ U_x \frac{\partial U_x}{\partial x} + U_y \frac{\partial U_x}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} \right) \\ U_x \frac{\partial U_y}{\partial x} + U_y \frac{\partial U_y}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \left(\frac{\partial^2 U_y}{\partial x^2} + \frac{\partial^2 U_y}{\partial y^2} \right) \end{cases} \quad (1)$$

Along with the following boundary conditions:

- No-slip condition at the surface: $U_x = U_y$ at $y = 0$;
- Uniform flow at infinity $U_x \rightarrow U_\infty$ when $y \rightarrow \infty$.

The above equation along with its boundary condition expresses a laminar boundary layer over a flat plate from which we can deduce Prandtl's equations given as

$$\begin{cases} \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} = 0 \\ U_x \frac{\partial U_x}{\partial x} + U_y \frac{\partial U_x}{\partial y} = \nu \frac{\partial^2 U_x}{\partial y^2} \\ \frac{\partial P}{\partial y} = 0 \end{cases} \quad (2)$$

Let's introduce the stream function Ψ defined by

$$\begin{cases} U_x = \frac{\partial \Psi}{\partial y} \\ U_y = -\frac{\partial \Psi}{\partial x} \end{cases} \quad (3)$$

Simple substitution implies

$$\frac{\partial \Psi}{\partial y} \frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x} \frac{\partial^2 \Psi}{\partial y^2} = \nu \frac{\partial^3 \Psi}{\partial y^3} \quad (4)$$

with the boundary conditions, which now become

- For any x at $y = 0$, we have: $\Psi = 0$ and $\frac{\partial \Psi}{\partial y} = 0$;
- When, $y \rightarrow \infty$ we have: $\frac{\partial \Psi}{\partial y} \rightarrow U_\infty$.

Next, we introduce a dimensionless variable η and function $f(\eta)$, respectively given by

$$\eta = \sqrt{\frac{U_\infty}{\nu x}} \cdot y \quad (5)$$

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and

$$\Psi = \sqrt{\nu U_\infty x} \cdot f(\eta) \quad (6)$$

All these transformations lead us to introduce the differential equation for the so-called Blasius problem

$$\frac{1}{2}f \cdot f'' + f''' = 0 \quad (7)$$

with the boundary conditions given as

$$\begin{cases} f = f' = 0 \text{ at } \eta = 0 \\ f' \rightarrow 1 \text{ as } \eta \rightarrow \infty \end{cases} \quad (8)$$

2. Blasius Problem and the Adomian Decomposition Method

As it can clearly be revealed from the exposition of the Blasius problem above, we have two main issues:

1. The first one is the problem formulation category, a nonlinear third order differential equation;
2. The second one is related to the absence of the value of $f''(0)$.

As it is well known [6], the Adomian decomposition method is efficient to overcome this kind of difficulty, however, its performance is limited and the approximated solution ends with a divergence [7].

On the other hand, to establish an iterative difference scheme of second order, we need at least a good accuracy for the value of initialization elements in order to start the iterative calculation.

In the sequel, we try to combine these two methods in order to elaborate a new single method that avoids the limitation of each method taken separately.

The Adomian decomposition method, when applied to the Blasius problem, leads us to consider:

$$u = f + Nu \quad (9)$$

with the following

$$\begin{cases} u = \sum_{n \geq 0} u_n \\ Nu = \sum_{n \geq 0} A_n \\ A_n = \frac{1}{n!} \frac{d^n}{d\eta^n} [Nu(\eta)]_{\eta=0}, \quad n \in \mathbb{N} \end{cases} \quad (10)$$

Where,

$$N = \frac{d^3}{d\eta^3} \quad (11)$$

According to [8] and [9], we have

$$\tilde{f}_3(\eta) = k \frac{1}{2} \eta^2 - k^2 \frac{1}{2.5!} \eta^5 + k^3 \frac{11}{4.8!} \eta^8 \quad (12)$$

with $k = f''(0) = Z$.

3. Difference Scheme of Second Order Accuracy for Blasius Problem

Here, we adopt a difference scheme [10] and [11] for solving the Blasius problem. Indeed, we can easily formulate this equation system as

$$\begin{cases} f_{i+1} = f_i + \Delta\eta \cdot f'_i \\ f'_{i+1} = f'_i + \Delta\eta \cdot f''_i \\ f''_{i+1} = f''_i + \Delta\eta \cdot f'''_i \\ f'''_{i+1} = \frac{1}{2} f'_{i+1} \cdot f''_{i+1} \end{cases} \quad (13)$$

In the third equation above, for the second order derivative, we use the following second order accuracy difference approximation

$$f''_{i+1} = \frac{f_{i+2} - 2f_{i+1} + f_i}{\Delta\eta^2} \quad (14)$$

The calculation algorithm is also presented diagrammatically, as shown in Fig. 1.

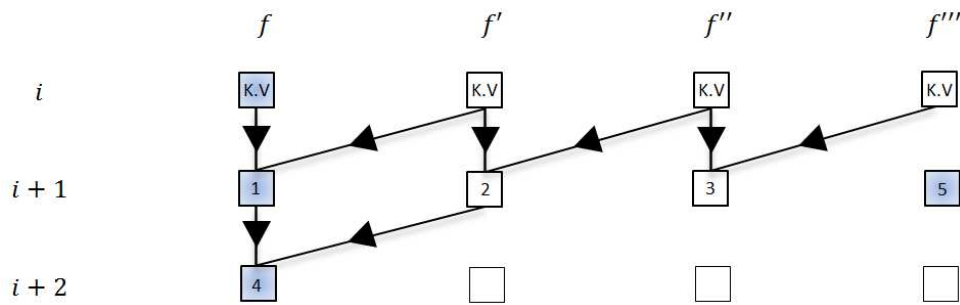


Fig.1: Illustration of the algorithm of calculation relative to the coupled method.

where:

- The numbers in the cases indicate the sequence of calculation from the *Known Values* (K.V);
- The arrows orientation emanating from the boxes of the top level indicate those needed for the calculation of the boxes of the down level while the iteration process is evolving;
- The blue boxes indicate those needed to calculate the 5th step.

4. The Coupled ADM-Finite Difference Method

As presented above, we will use a coupling between the Adomian decomposition method and a

finite difference scheme for the numerical resolution of the Blasius problem.

Hence, the ADM will provide the first elements, given the accuracy of the results so obtained. This operation concerns only the calculation of the values of $f(\eta)$ for $i \in \llbracket 1, 2 \rrbracket$. It is up to the difference scheme that was presented previously to take over and generate the rest of all values.

This coupled method, when applied to our case, gives the following numerical results after calculations (Table 1 and Fig. 2).

Table 1: Results of the numerical calculation for f and its derivatives.

η	f	f'	f''	f'''
0	0	0	0,332	0
0,1	0,00166	0,0332	0,332	-0,00027555
0,2	0,00663985	0,0664	0,33197245	-0,00055116
0,3	0,01327985	0,09959724	0,33191733	-0,00220427
0,4	0,02323958	0,13278898	0,3316969	-0,00385681
0,5	0,03651848	0,16595867	0,33131122	-0,00605653
0,6	0,05311434	0,19908979	0,33070557	-0,00879869
0,7	0,07302332	0,23216035	0,3298257	-0,01207461
0,8	0,09623936	0,26514292	0,32861824	-0,01587111
0,9	0,12275365	0,29800474	0,32703113	-0,02016954
1	0,15255412	0,33070785	0,32501417	-0,02494497

The above results are presented synthetically in Fig. 2.

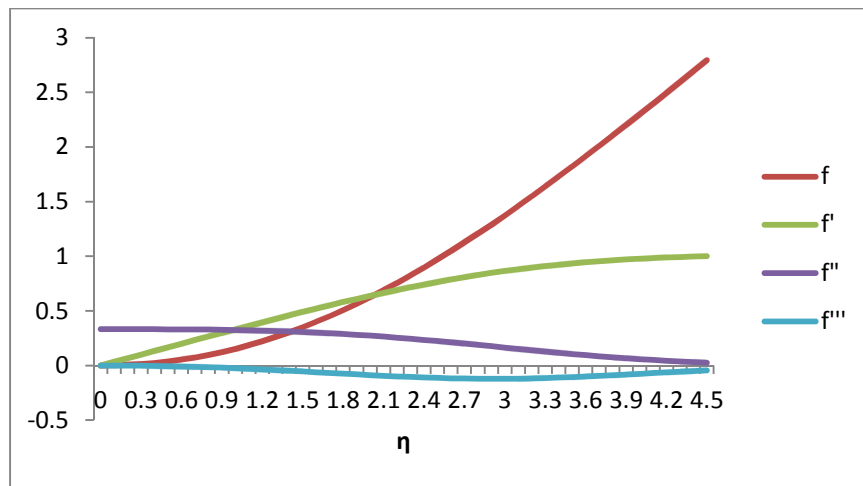


Fig.2: The graphical representations of f and its derivatives.

Through this work, we were able to elaborate a coupling method based on two numerical methods: the Adomian decomposition and the finite difference. The results so obtained correspond to the boundary conditions, especially $f' \rightarrow 1$ for fairly significant values of η . The graphic representation feature is similar to those commonly provided by others methods. We can say that this combination used for the numerical solution of the Blasius problem in fluid dynamics is an efficient algorithm, which can offset the limitation of each method taken separately.

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