

Linear Stability Analysis of Fluid Flow between Two Parallel Porous Stationary Plates with Small Suction and Injection

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In this work, the linear stability of the viscous incompressible fluid flow between two parallel horizontal porous stationary plates with the assumption that there is a small constant suction at upper plate and a small constant injection at the lower plate is studied. The Navier-Stokes and continuous equations are reduced to an equation modified by the suction Reynolds number, which we call modified Orr-Sommerfeld equation. This equation is rewritten as an eigenvalue problem and is solved numerically using Matlab 7.0 (windows version). The effect of small suction or small injection Reynolds number on the linear stability fluid flow is discussed.

1. Introduction

The flow through porous boundaries is of great importance both in technological as well as biophysical flows. Examples of this are found in soil mechanics, aviation industry, transpiration cooling, food preservation, cosmetic industry, blood flow and artificial dialysis. A large number of theoretical investigations dealing with steady incompressible laminar flow with either injection or suction at the boundaries ([1,2]) have appeared during the few decades. Several authors [3] have studied the steady laminar flow of an incompressible viscous fluid in a two-dimensional channel with parallel porous walls. Rioual et al. [4] studied the power balance of a flat plate used as an airplane wing and found that there is an optimum speed of suction that reduces drag and weaken energy consumption. Recent experimental studies show that suction damps growth of disturbances induced by free stream turbulence and transition is delayed/prevented [5]. Aspiration may also be used as a tool to induce transition, instead of the delay, if for example the wall material chosen is not able to provide continuous suction [6] or if it is applied non-uniformly [7] and [8].

In this work, the flow of the incompressible viscous fluid between two parallel horizontal stationary porous plates with the assumption that there is a constant small suction at upper plate and a constant small injection at the lower plate is considered. Speeds suction and injection are

assumed to be uniform and same standard V_ω^* (see Fig. 1 below) [3]. The objective is to study the effect of number Re_ω of extradvective term introduced in an Orr-Sommerfeld equation by small suction velocity on the stability of the fluid flow. Such attempt has been made earlier by Davidsson and Gustavsson [1,9] for the case of boundary layers (a flat plat-law) with uniform wall suction by normalizing all components of the velocity with the free stream velocity U^* . Before Davidsson and Gustavsson, Bansal and Bhatnagar [10], and Soundalgekar and Divekar [11]) studied the flow quilt with suction at the stationary plate and laminar slip-flow through uniform circular pipe with small suction. All these studies confirmed the stabilizing effect of small suction on the flow.

The paper is organized as follows. In Sec. 2, the modified Orr-Sommerfeld equation governing the stability analysis of two parallel horizontal porous stationary plates, the Poiseuille flow, is checked. In Sec. 3, an analysis of small suction (small injection) Reynolds number Re_ω stabilizing effect is investigated. The conclusion is presented in Sec. 4.

2. Modified Orr-Sommerfeld Equation

The motion of a fluid is governed by various conservation laws such as the law of conservation of mass and the law of conservation of momentum. The momentum of the fluid changes due to various forces acting upon it and is governed by Newton's second law of motion. Fluids at low velocity as compared to the speed of sound can be considered incompressible, i.e., the density of the fluid is constant. The stress acting on a fluid element may

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be linearly proportional to the local pressure and velocity gradient. For an incompressible Newtonian fluid the momentum and conservation equations yield

$$\frac{\partial \tilde{u}_i^*}{\partial \tilde{t}^*} + \tilde{u}_j^* \frac{\partial \tilde{u}_i^*}{\partial \tilde{x}_j^*} = -\frac{1}{\rho} \frac{\partial \tilde{p}^*}{\partial \tilde{x}_i^*} + \mathcal{V} \frac{\partial^2 \tilde{u}_i^*}{\partial \tilde{x}_j^{*2}} \quad (1)$$

$$\frac{\partial \tilde{u}_i^*}{\partial \tilde{x}_i^*} = 0 \quad (2)$$

when written by Einstein's summation convention. These are the Navier-Stokes (N-S) and conservation of mass equations, respectively. Here $\tilde{x}_i^*$ and $\tilde{u}_i^*$ are the i^{th} component of space and velocity vectors, as commonly seen in the literature denoted by $(\tilde{x}^*, \tilde{y}^*, \tilde{z}^*)$ and $(\tilde{u}^*, \tilde{v}^*, \tilde{w}^*)$ in the stream wise, wall-normal and span wise directions, respectively. Also, $\tilde{p}^*$ is the pressure, ρ is the density and $\mathcal{V}$ is the kinematical viscosity of the fluid.

We considered the viscous and incompressible fluid flow between two parallel horizontal stationary porous flat plates under a constant gradient of pressure with small wall suction and injection showing in Fig.1 below [3].

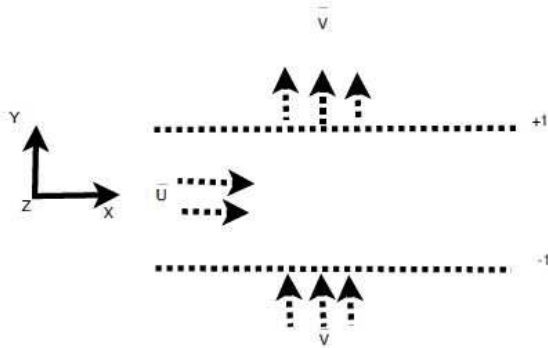


Fig.1: Poiseuille porous parallel flat plates flow.

The coordinates (stream-wise x , the wall-normal y and span-wise z) are scaled with the length scale h^* (distance between the two walls). The stream-wise and span-wise velocities u and w , respectively, are scaled with the stream-wise free stream velocity U^* while the wall-normal velocity v is scaled with the characteristic velocity of suction and injection V_ω^* . The pressure p is scaled with ρU^{*2} and the time t with $\frac{h^*}{U^*}$. The Reynolds' numbers used here are hydrodynamic Reynolds number $Re = \frac{U^* h^*}{\nu}$ and suction Reynolds number, $Re_\omega = \frac{V_\omega^* h^*}{\nu}$, where ν is the kinematical viscosity. We want to study the linear stability of the flow at high hydrodynamic

Reynolds number Re and small suction Reynolds Re_ω ($0 \leq Re_\omega \leq 1$) numbers flow. The non-dimensional Navier-Stokes and continuous (conservation of mass) linearized Eqns. (1) and (2), where the variables are normalized as above, take the following forms

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{Re_\omega}{Re} \tilde{v} \frac{\partial \tilde{u}}{\partial \tilde{y}} + \tilde{w} \frac{\partial \tilde{u}}{\partial \tilde{z}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} \\ + \frac{1}{Re} \nabla^2 \tilde{u} \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{u} \frac{\partial \tilde{v}}{\partial \tilde{x}} + \frac{Re_\omega}{Re} \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{y}} + \tilde{w} \frac{\partial \tilde{v}}{\partial \tilde{z}} = -\frac{Re}{Re_\omega} \frac{\partial \tilde{p}}{\partial \tilde{y}} \\ + \frac{1}{Re} \nabla^2 \tilde{v} \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{\partial \tilde{w}}{\partial \tilde{t}} + \tilde{u} \frac{\partial \tilde{w}}{\partial \tilde{x}} + \frac{Re_\omega}{Re} \tilde{v} \frac{\partial \tilde{w}}{\partial \tilde{y}} + \tilde{w} \frac{\partial \tilde{w}}{\partial \tilde{z}} = -\frac{\partial \tilde{p}}{\partial \tilde{z}} \\ + \frac{1}{Re} \nabla^2 \tilde{w} \end{aligned} \quad (5)$$

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{Re_\omega}{Re} \frac{\partial \tilde{v}}{\partial \tilde{y}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0 \quad (6)$$

For the stability analysis, the flow is decomposed into the mean flow and the disturbance according to

$$\tilde{u}_i = U_i(r) + u_i(r, t) \quad (7)$$

$$\tilde{p}(r, t) = P(r) + p(r, t) \quad (8)$$

We take the dimensional base flow for small suction and injection [3]

$$U(y) = U^* \left(1 - \frac{y^2}{h^{*2}}\right) \quad (9)$$

$$V = V_\omega^* \quad (10)$$

$$W = 0 \quad (11)$$

By scaling these velocities as above, we obtain with $h^* = \pm 1$ the no-dimensional base flow

$$U(y) = (1 - y^2) \quad (12)$$

$$V = 1 \quad (13)$$

$$W = 0 \quad (14)$$

To obtain the stability equations for 3-dimensional spatial evolution, we take the dependency on time disturbances

$$(u(x, y, z, t); v(x, y, z, t); w(x, y, z, t); p(x, y, z, t)) \quad (15)$$

which are scaled in the same way as above.

The pressure terms can be eliminated from Navier- Stokes equations. For such a mean profile (base flow), the divergence of Navier-Stokes and continuity equations gives

$$\nabla^2 p = -2 \frac{Re_\omega}{Re} \frac{\partial U}{\partial y} \frac{\partial v}{\partial x} \quad (16)$$

The linearization of Eqns. (4) and (16) gives

$$\frac{\partial}{\partial t} \nabla^2 v + U \frac{\partial}{\partial x} \nabla^2 v + \frac{Re_\omega}{Re} \frac{\partial}{\partial y} \nabla^2 v - \frac{d^2 U}{dy^2} \frac{\partial v}{\partial x} = \frac{1}{Re} \nabla^4 v \quad (17)$$

The disturbances are taken to be periodic in time in both the stream-wise and span-wise directions, which allow us to assume solutions of the form

$$f(x, y, z, t) = \hat{f}(y) e^{i(\alpha x + \beta z - \omega t)} \quad (18)$$

Where, f represents either one of the disturbances u, v, w or p and $\hat{f}$ the amplitude function, $k, \alpha = k \cos(\theta)$ and $\beta = k \sin(\theta)$ are the wave numbers, $\omega = \alpha c$, the pulsation of the wave with $i^2 = -1, \theta = (\vec{k}_x, \vec{k})$, $c = c_r + i c_i$ wave velocity, which is taken to be complex, and α and β are real because of temporal stability analysis consideration.

Then with Eqn. (18), Eqn. (17) becomes

$$i\alpha[(U - c)(D^2 - k^2) - U'']\hat{v} = \frac{Re_\omega}{Re} D(D^2 - k^2)\hat{v} + \frac{1}{Re} (D^2 - k^2)^2 \hat{v} \quad (19)$$

Where, $D = \frac{d}{dy}$, with boundary conditions for all $(x, \pm 1, z, t)$

$$\begin{cases} \hat{v}(\pm 1) = 1 \\ \hat{v}'(\pm 1) = 0 \end{cases} \quad (20)$$

Taking

$$v_p(x, y, z, t) = v(x, y, z, t) - 1 \quad (21)$$

Eqn. (19) and the boundary conditions take the forms

$$\begin{aligned} & \left[\left(U - i\alpha^{-1} \frac{Re_\omega}{Re} D \right) (D^2 - k^2) - U'' \right. \\ & \quad \left. - i\alpha^{-1} \frac{1}{Re} (D^2 - k^2)^2 \right] \hat{v}_p \\ & = c(D^2 - k^2) \hat{v}_p \end{aligned} \quad (22)$$

$$\begin{cases} \hat{v}_p(\pm 1) = 0 \\ \hat{v}_p'(\pm 1) = 0 \end{cases} \quad (23)$$

Eqn. (22) is a flow equation modified by suction Reynolds number Re_ω (or the speed of suction and injection), which we call modified Orr-Sommerfeld equation, rewritten as an eigenvalue problem, where c is the eigenvalue, and $\hat{v}_p$ the eigenfunction.

$\left[\left(U - i\alpha^{-1} \frac{Re_\omega}{Re} D \right) (D^2 - k^2) - U'' - i\alpha^{-1} \frac{1}{Re} (D^2 - k^2)^2 \right]$ and $(D^2 - k^2)$ are the operators.

3. Stability Analysis

We consider now 3-dimensional disturbances and use a temporal stability analysis as mentioned above. With c complex as we have defined above, when $c_i < 0$ we have stability and for $c_i = 0$ we have neutral stability, and elsewhere we have instability. We employ Matlab 7.0 (windows version) in all our numerical computations to find the eigenvalues. The Poiseuille parallel horizontal stationary porous plates flow with the basic velocity profile

$$U = ((1 - y^2), 1, 0) \quad (24)$$

for Re_ω small (i. e., small suction) is considered [3]. The eigenvalue problem (Eqn. (22)) is solved numerically with the suitable boundary conditions. The solutions are found in a layer bounded at $y = \pm 1$ with $U(\pm 1) = (0, 1, 0)$. The results of calculations are presented in the figures below. We present the figures related to the eigenvalue problem (Eqn. (22)).

For all these figures the black, red, green and blue colors are, respectively, for, $Re_\omega = 0, Re_\omega = 0.5, Re_\omega = 0.75, Re_\omega = 1$ and the yellow color is for $c_i = 0$.

In each group of four figures, the first one is figure a), the second is figure b) the third is figure c) and the fourth is figure d).

For a fixed $k=1$, we get figure (2) of c_i vs. R_e for sequential values of θ . Figure a) for $\theta = 0$, figure b) for $\theta = 0.1\pi$, figure c) for $\theta = 0.2\pi$ and figure d) for $\theta = 0.3\pi$.

For a fixed $k=1.02$, we get figure (3) of c_i vs. R_e for sequential values of θ . Figure a) for $\theta = 0$, figure b) for $\theta = 0.1\pi$, figure c) for $\theta = 0.2\pi$ and figure d) for $\theta = 0.3\pi$.

For a fixed $k=2$, we get figure (4) of c_i vs. R_e for sequential values of θ . Figure a) for $\theta = 0$, figure b) for $\theta = 0.1\pi$, figure c) for $\theta = 0.2\pi$ and figure d) for $\theta = 0.3\pi$.

For a fixed $k=3$, we get Fig. (5) of c_i vs. R_e for sequential values of θ . Figure a) for $\theta = 0$, figure b) for $\theta = 0.1\pi$, figure c) for $\theta = 0.2\pi$ and figure d) for $\theta = 0.3\pi$.

Through Figs. 2 and 3, it is easy to see that the stability increases when $R_{e\omega}$ increases for all θ because for any family of curves of these figures the slope falls when the suction Reynolds' number increases. The Reynolds' critical number R_{ec} for which we have the transition becomes important when $R_{e\omega}$ increases (see Table 1), which confirms that the wall small suction and injection have a stabilizing effect on the viscous incompressible fluid flow. Because of $R_{e\omega}$ influences, we said that

normalizing with the characteristic velocity of suction/ injection is necessary for a perfect command of the field of stability. In particular for ($R_{e\omega} = 0, k = 1.02$, without suction and injection, Fig. 3 black curve (a)) we find $R_{ec} = 5772.17$, which corresponds exactly to the critical value given by classical linear theory for a plane-Poiseuille flow without suction and injection.

For $R_{e\omega} = 1$, Eqn. (22) is identical for the Eqn. (2.28) found by Davidsson and Gustavsson [1]. For this value of $R_{e\omega}$ in the cases of Figs. 2 and 3 the field of stability is more than the other small suction Reynolds number for all θ .

Note that for large value k the case of the Figs. 4 and 5, there is stability without transition for all value of θ and $R_{e\omega}$ but when the small suction $R_{e\omega}$ increases the field of stability decreases. We can also say that k influences the stability of the flow i.e., the growth of the wave number induces also the stability of this flow. We therefore conclude that in two parallel horizontal stationary porous plates viscous and incompressible fluid flow, with small suction and small injection, the small suction Reynolds number stabilize the flow for small wave number but the high wave number effect is important than small suction Reynolds' number on the fluid flow stability.

Table 1: Critical values

		$k = 1.00$	$k = 1.02$
θ	$R_{e\omega}$	R_{ec}	R_{ec}
0.00π	0.00	5815.00	5772.17
0.00π	0.50	5822.80	5780.83
0.00π	0.75	5832.33	5790.88
0.00π	1.00	5845.56	5805.59
0.10π	0.00	6114.17	6069.68
0.10π	0.50	6122.13	6078.38
0.10π	0.75	6132.18	6089.00
0.10π	1.00	6146.03	6104.49
0.20π	0.00	7187.57	7135.25
0.20π	0.50	7197.25	7145.54
0.20π	0.75	7208.53	7158.28
0.20π	1.00	7225.20	7176.30
0.30π	0.00	9892.84	9820.66
0.30π	0.50	9905.71	9834.88
0.30π	0.75	9921.64	9877.03
0.30π	1.00	9944.31	9877.03

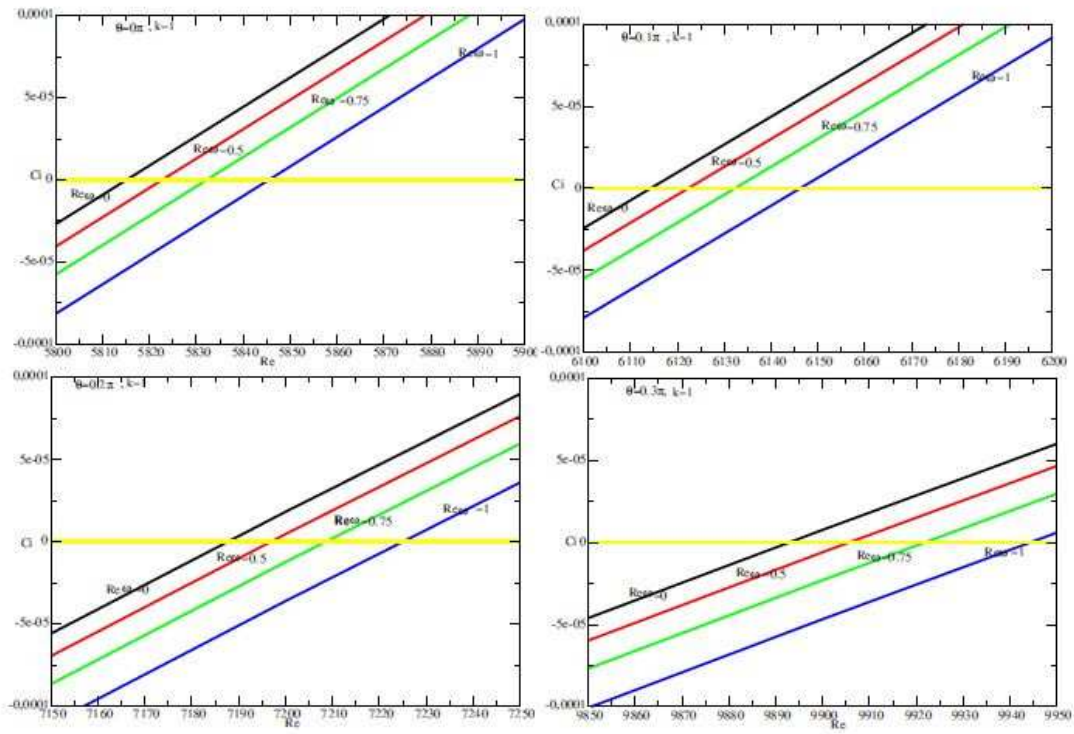


Fig.2: Growth rate c_i vs. Re for $k=1$.

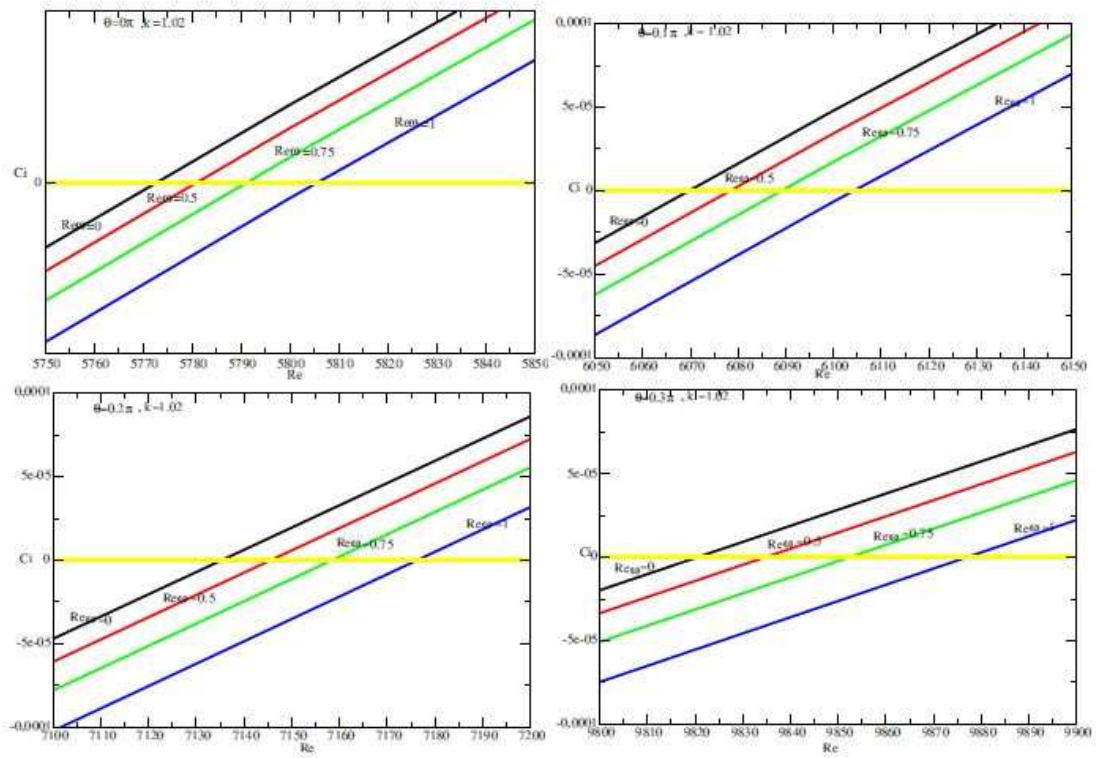
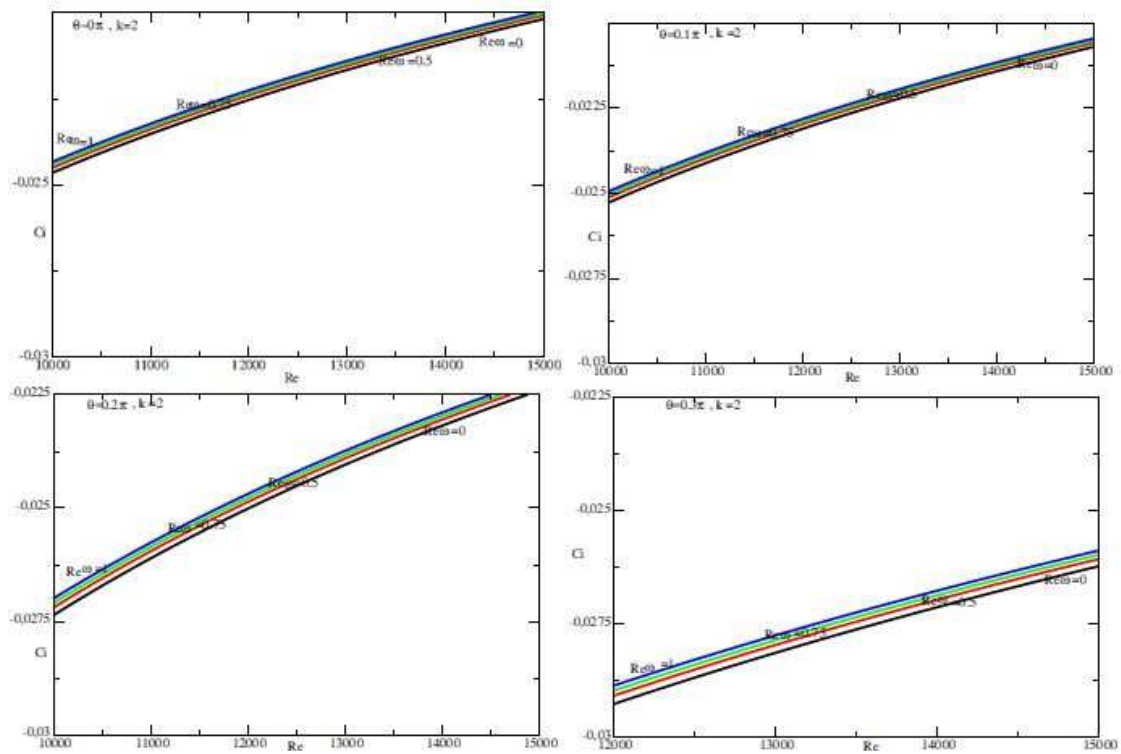
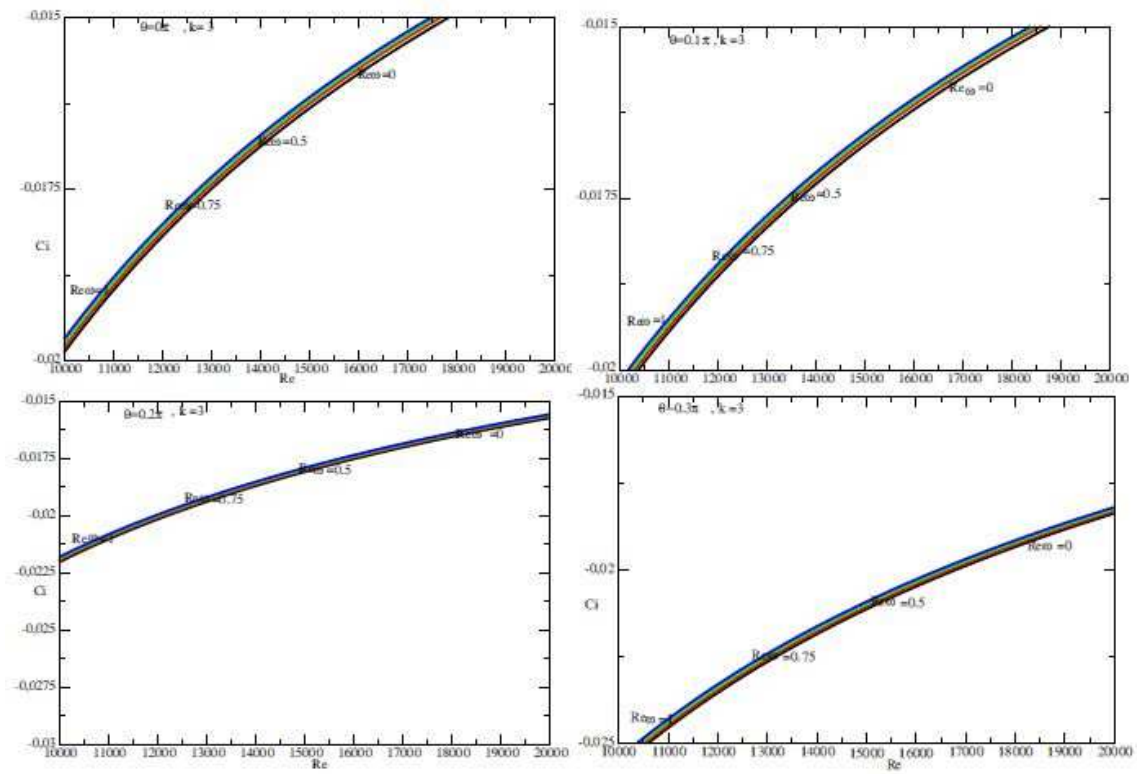


Fig.3: Growth rate c_i vs. Re for $k=1.02$.

Fig.4: Growth rate c_i vs. R_e for $k=2$.Fig.5: Growth rate c_i vs. R_e for $k=3$.

4. Conclusion

In this paper, we have investigated the effect of small suction Reynolds number on the stability of the fluid flow between two parallel horizontal stationary porous plates. We have shown that the instability of the perturbed flow is governed by an equation named modified Orr-Sommerfeld equation. Moreover, we find that the normalization of the wall-normal velocity with characteristic small suction (or small injection) velocity is very important for a perfect command of fluid flow stability analysis. We noticed that $k = 1.02$ and $Re_\omega = 0$, we find $Re_{cc} = 5772.17$, which corresponds exactly to the critical value given by classical linear theory for a plane-Poiseuille flow without suction and injection. We noticed also that the high wave number stabilizes more than small suction Reynolds number.

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