

Effects of Magnetic Field and Helical Force on the Onset of Rayleigh-Bénard Convection with Free-Free Boundaries

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The combined effect of both the magnetic field and helical force on the onset of stationary and oscillatory convections in a horizontal electrically conducting fluid layer heated from below with free-free boundary conditions is examined through the application of the linear stability analysis. This analysis allowed us to obtain the condition for the formation of a single large-scale structure and the condition for the appearance of two cells at marginal stability for the oscillatory convection.

1. Introduction

The consideration of the problem of thermal instability in electrically conducting fluids in the presence of a magnetic field in juxtaposition with a similar consideration of the effects of rotation brings out some striking similarities between the effects of rotation and magnetic field. They both inhibit the onset of instability and they both elongate cells which appear at marginal stability. In the presence of magnetic field or in rotating systems such as planetary atmospheres, the turbulent convection motion is known to become helical [1,2]. In modern physics, small-scale helical turbulence is known to have a number of special features, such as under certain conditions it is capable of intensifying and sustaining large-scale vortex disturbances by means of energy transfer from small to large scales. For the first time in magneto-hydrodynamic (MHD), Steenbeck, Krause and Rädler [3] have discovered the generating properties of small-scale helical turbulence. This phenomenon is called the alpha-effect. This discovery has favored a vigorous development of MHD-dynamo theory. Another important incentive was the formal similarity of equations describing the magnetic field in a moving electrically-conducting medium and vorticity in non-conducting fluids. The hydro-dynamical α -effect in compressible [4] and incompressible [5] fluids was discovered by Moiseev and al. Based on the results obtained in [4], Moiseev et al. [6] proposed a phys-

ical mechanism for enhancing large-scale vortex disturbances in the atmosphere using the energy of small-scale helical turbulence. In Levina et al. [9], it was proposed that in a convective system, the special force with the structure of α -term provides excitation of a large-scale instability within the frame work of hydro-dynamical α -effect model. Thus, it has been demonstrated that the action of this model force, also called helical force within the Rayleigh-Bénard convection, may initiate a new type of instability, namely the helical-vortex instability. According to [10], for three different types of boundary conditions, the linear stability analysis showed that the helical force decreases the threshold of instability.

The multiplicity of its control parameters makes the Rayleigh-Bénard convection interesting in the presence of magnetic field for the study of hydrodynamical stability. Chandrasekhar [11] has shown that the convection can set in as stationary or oscillatory motions.

In this work, the classical linear stability analysis is applied to investigate the combined effect of both the magnetic field and the helical force on the onset of stationary and oscillatory convections with free boundary conditions. The condition for obtaining the formation of one large-scale structure is derived and the condition for obtaining the apparition of two cells at marginal stability is obtained.

2. Mathematical Formulation of the Problem

Let us consider an infinite horizontal layer of an electrically conducting fluid upon which is

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impressed a uniform magnetic field in the presence of helical force.

In the Boussinesq approximation, we have:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{\vec{\nabla} p}{\rho_0} + \frac{\mu}{4\pi\rho} (\text{curl} \vec{H}) \times \vec{H} + \nu \nabla^2 \vec{v} - \alpha \Delta T \vec{g} + a\Omega d \vec{f} \quad (1)$$

$$\frac{\partial \vec{H}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{H} = (\vec{H} \cdot \vec{\nabla}) \vec{v} + \eta \nabla^2 \vec{H} \quad (2)$$

$$\frac{\partial T}{\partial t} + (\vec{v} \cdot \vec{\nabla}) T = \kappa \nabla^2 T \quad (3)$$

$$\vec{\nabla} \cdot \vec{H} = 0 \quad \text{and} \quad \vec{\nabla} \cdot \vec{v} = 0 \quad (4)$$

$$\vec{f} = \vec{e}(\text{curl}(\vec{v}))_z - \frac{\partial(\vec{e} \times \vec{v})}{\partial z}; \quad \vec{e} = (0,0,1)$$

Where, $\vec{v}$ is fluid particle velocity vector, p is pressure, T is temperature, $\vec{f}$ is helical force, $\vec{e}$ is the unit vector along the vertical axis, $\vec{H}$ is magnetic field, μ is magnetic permeability, ρ is density of fluid particle, ρ_0 is density of fluid particle at temperature T_0 , ν is the kinematic viscosity, η is resistivity, κ is thermal diffusivity of the fluid particle, $\vec{\Omega}$ is angular rotation vector, d is layer height, k is heat conductivity of the fluid particle, C_v is specific heat capacity of the fluid particle, $\kappa = \frac{k}{\rho_0 C_v}$, $a\Omega d$ is the amplitude of model force, α is coefficient of volume expansion, ΔT is difference temperature between the boundaries of the layer, the second term on the right-hand side of Eqn. (1) is the Lorentz force, which incorporates the effect of the magnetic field on the motion, and $\vec{g}$ is acceleration due to gravity.

3. Linear Stability Analysis

Following the classical lines of linear stability theory as presented by Chandrasekhar [11], the linearized governing equations can be written as

$$(D^2 - k^2)(D^2 - k^2 - \sigma)W = \frac{g\alpha k^2 d^2 \theta}{\nu} - \frac{\mu H d}{4\pi\rho\nu} D(D^2 - k^2)K + \frac{a\Omega d^3}{\nu} (k^2 + D^2)Z \quad (5)$$

$$(D^2 - k^2 - \sigma)Z = -\frac{a\Omega d}{\nu} D^2 W - \frac{\mu H d}{4\pi\rho\nu} DX \quad (6)$$

$$(D^2 - k^2 - P_m\sigma)K = -\frac{Hd}{\eta} DW \quad (7)$$

$$(D^2 - k^2 - P_m\sigma)X = -\frac{Hd}{\eta} DZ \quad (8)$$

$$(D^2 - k^2 - P_r\sigma)\theta = -\frac{\beta d^2}{K} W \quad (9)$$

Where, W , Z , K , X , and θ are the amplitude of vertical velocity, vertical vorticity, vertical magnetic field, vertical current density, and temperature, respectively. Here, $P_r = \frac{\nu}{\kappa}$ is the Prandtl number and $P_m = \frac{\nu}{\eta}$ is the magnetic Prandtl number.

The operator $D = \frac{d}{dz}$ denotes differentiation with respect to the vertical coordinate, z , k is the dimensionless wave number and σ the stability parameter. The stability parameter σ is in general a complex variable denoted by $\sigma = \gamma + i\omega$, where γ is the growth rate of the instability and ω is the frequency. If, $\gamma > 0$, the disturbance grows and the system becomes unstable. Whereas, if $\gamma < 0$ the disturbance decays and the system becomes stable. When $\gamma = 0$, the instability of the system at the marginal state sets in as stationary motions provided $\omega = 0$, or as oscillatory motions provided $\omega \neq 0$.

Eliminating K , Z and θ from these equations and after a few algebraic manipulations, we get the dimensionless governing equation

$$\begin{aligned} & (D^2 - k^2 - P_r\sigma)[(D^2 - k^2)(D^2 - k^2 - \sigma)(D^2 - k^2 - P_m\sigma) - QD^2]^2 + S^2 D^2 (D^2 + k^2)(D^2 - k^2 - P_m\sigma)^2]W \\ & = -R_a k^2 [(D^2 - k^2 - \sigma)(D^2 - k^2 - P_m\sigma) - QD^2](D^2 - k^2 - P_m\sigma)W \end{aligned} \quad (10)$$

Where, $S = \frac{a\Omega d^2}{\nu}$ is a dimensionless parameter that characterizes the intensity of force $\vec{f}$, $R_a = \frac{g\alpha\beta d^4}{\nu\kappa}$ is dimensionless Rayleigh number, and, $Q = \frac{\mu H^2 d^2}{4\pi\rho\nu\eta}$, is a non-dimensional number. This number, Q , plays for problems involving magnetic field the same role that Taylor number T_a plays for problems involving rotation.

In this work, we consider the free-free boundary condition. Hence, W and all of its even derivatives vanish at $z=0$ and $z=1$. The solution of Eqn. (10) is of the form $W = \sin(\pi z)$.

Substituting $W = \sin(\pi z)$ and $\sigma = i\omega$ in Eqn. (10), one gets:

$$R_a = \frac{(\pi^2 + k^2 + i\omega P_r) \left\{ \frac{(\pi^2 + k^2) [(\pi^2 + k^2 + i\omega)(\pi^2 + k^2 + i\omega P_m) + Q\pi^2]^2 + S^2 \pi^2 (k^2 - \pi^2) (\pi^2 + k^2 + i\omega P_m)^2}{k^2 (\pi^2 + k^2 + i\omega P_m) [(\pi^2 + k^2 + i\omega)(\pi^2 + k^2 + i\omega P_m) + Q\pi^2]} \right\}}{k^2 (\pi^2 + k^2 + i\omega P_m) [(\pi^2 + k^2 + i\omega)(\pi^2 + k^2 + i\omega P_m) + Q\pi^2]} \quad (11)$$

When the instability sets in as stationary convection, the marginal state will be characterized by $\omega = 0$.

Substituting $\omega = 0$ in Eqn. (11), one obtains

$$R_{as} = \frac{(\pi^2 + k_{sc}^2) \left\{ [(\pi^2 + k_{sc}^2)^2 + Q\pi^2]^2 - S^2 \pi^2 (\pi^2 - k_{sc}^2) (\pi^2 + k_{sc}^2) \right\}}{k_{sc}^2 [(\pi^2 + k_{sc}^2)^2 + Q\pi^2]} \quad (12)$$

Here R_{as} is the value of the Rayleigh number for stationary convection. The critical value of R_{as} is obtained for $k_s = k_{sc}$ satisfying the following:

$$\begin{aligned} & -(\pi^2 + k_{sc}^2)^7 + 3k_{sc}^2 (\pi^2 + k_{sc}^2)^6 - 3Q\pi^2 (\pi^2 + k_{sc}^2)^5 \\ & + 7Q\pi^2 k_{sc}^2 (\pi^2 + k_{sc}^2)^4 + 5Q^2 \pi^4 (\pi^2 + k_{sc}^2)^2 k_{sc}^2 \\ & + S^2 \pi^4 (\pi^2 + k_{sc}^2)^4 + QS^2 \pi^4 k_{sc}^2 (\pi^2 + k_{sc}^2) (3k_{sc}^2 - \pi^2) \\ & - Q^3 \pi^8 - 3Q^2 \pi^4 (\pi^2 + k_{sc}^2)^3 - QS^2 \pi^4 (k_{sc}^2 - \pi^2) \\ & (\pi^2 + k_{sc}^2)^2 = 0 \end{aligned} \quad (13)$$

$$\begin{aligned} R_{ao} &= \frac{\pi^2 + k^2}{k^2} \left\{ [(\pi^2 + k^2)^2 - P_r \omega^2] + \frac{Q\pi^2 [(\pi^2 + k^2)^2 + P_r P_m \omega^2]}{(\pi^2 + k^2)^2 + \omega^2 P_m^2} \right. \\ &+ \left. \frac{S^2 \pi^2 (k^2 - \pi^2)}{k^2 + \pi^2} \frac{[(\pi^2 + k^2)^2 - P_m \omega^2 + Q\pi^2] [(\pi^2 + k^2)^2 - P_r P_m \omega^2] + \omega^2 (\pi^2 + k^2)^2 (P_m + P_r) (P_m + 1)}{[(\pi^2 + k^2)^2 - P_m \omega^2 + Q\pi^2]^2 + \omega^2 (\pi^2 + k^2)^2 (P_m + 1)^2} \right\} \end{aligned} \quad (15)$$

And

$$\begin{aligned} & \frac{S^2 \pi^2 (k^2 - \pi^2)}{k^2 + \pi^2} \frac{(\pi^2 + k^2)^2 (P_r - 1) + Q\pi^2 (P_m + P_r) + \omega^2 P_m^2 (P_r - 1)}{[(\pi^2 + k^2)^2 - P_m \omega^2 + Q\pi^2]^2 + \omega^2 (\pi^2 + k^2)^2 (P_m + 1)^2} + \\ & \frac{Q\pi^2 (P_r - P_m)}{(\pi^2 + k^2)^2 + \omega^2 P_m^2} + 1 + P_r = 0 \end{aligned} \quad (16)$$

When $S = 0$ in Eqns. (15,16), we recover the results obtained by Chandrasekhar [11].

When $Q = 0$ in Eqn. (13), we recover the results obtained by Essoun et al. [13] when $T_a = 0$ and $k_s = k_{sc}$.

Eqn. (13) can be written as

$$\begin{aligned} & 4 \left(-1 + \frac{(S^2 - 2Q)}{\pi^2} - \frac{Q^2}{\pi^4} \right) \left(\frac{k_{sc}}{\pi} \right)^2 \\ & + \left(-3 + \frac{6S^2 - 2Q}{\pi^2} + \frac{Q(Q + S^2)}{\pi^4} \right) \left(\frac{k_{sc}}{\pi} \right)^4 \\ & + \left(10 + \frac{4(3Q + S^2)}{\pi^2} + \frac{2Q^2(1 + S^2)}{\pi^4} \right) \left(\frac{k_{sc}}{\pi} \right)^6 \\ & + \left(25 + \frac{S^2 + 13Q}{\pi^2} \right) \left(\frac{k_{sc}}{\pi} \right)^8 + \left(24 + \frac{4Q}{\pi^2} \right) \left(\frac{k_{sc}}{\pi} \right)^{10} \\ & + 11 \left(\frac{k_{sc}}{\pi} \right)^{12} + 2 \left(\frac{k_{sc}}{\pi} \right)^{14} \\ & = 1 + \frac{3Q}{\pi^2} + \frac{3Q^2 - QS^2 - S^2 \pi^2}{\pi^4} + \frac{Q^3}{\pi^6} \end{aligned} \quad (14)$$

When $S = 0$ in Eqn. (12), we recover the results obtained by Chandrasekhar, while the case $Q=0$ corresponds to those obtained by Levina et al. [10].

For the oscillatory convection ($\omega \neq 0$) and from Eqn. (11), R_a can be shown to be complex. However, the physical meaning of R_a requires it to be real. The condition that R_a is real implies that the imaginary part of Eqn. (11) is zero, i.e.

4. Results and Discussion

From Eqn. (14), when $S > \pi$ and $(\pi^2 + Q)^2 - S^2 \pi^2 = 0$, the critical wave number for the onset of stationary convection tends to zero. In other words, when S approaches some critical value $S_{cr} = \pi + \frac{Q}{\pi}$ or Q approaches $Q_{cr} = \pi(S - \pi)$, the critical wave number k_{sc} tends to zero while the critical Rayleigh number R_{as} tends to a limiting value $R_{as}^* = 4\pi^4$. Here, R_{as}^* does not depend on S and Q . $R_{as}^* = 4\pi^4$ corresponds to the case where the magnetic field is absent.

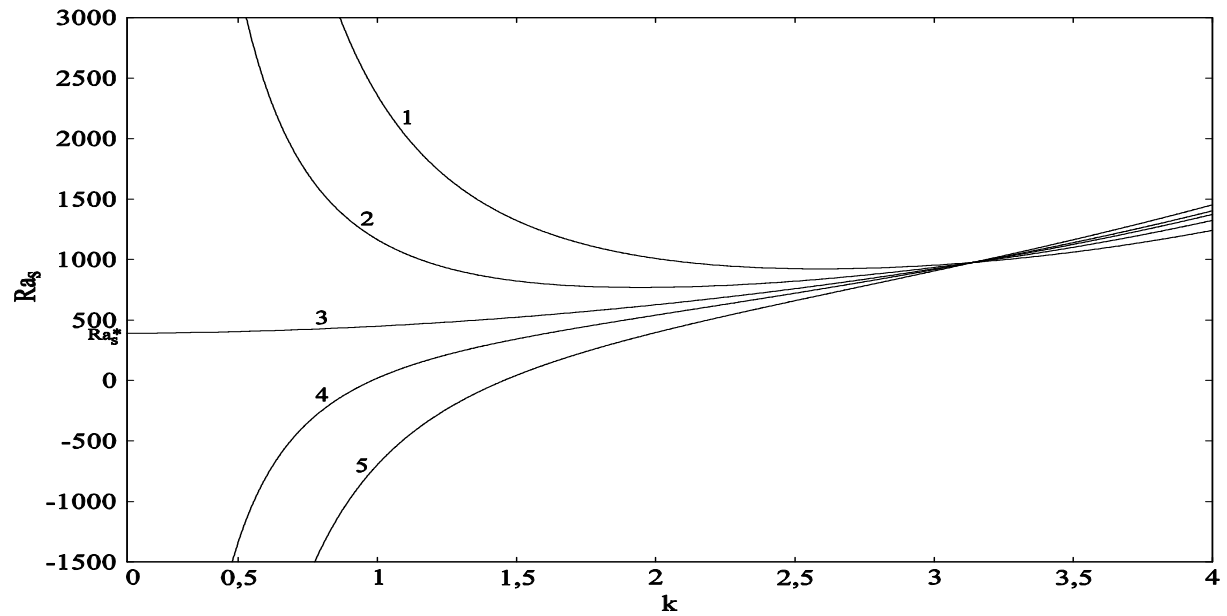


Fig.1: Neutral stability curves for the onset of stationary convection in the case $Q=10$ and for several values of S : (1) $S=0$; (2) $S=5$; (3) $S=6.32469$; (4) $S=7$ and (5) $S=8$.

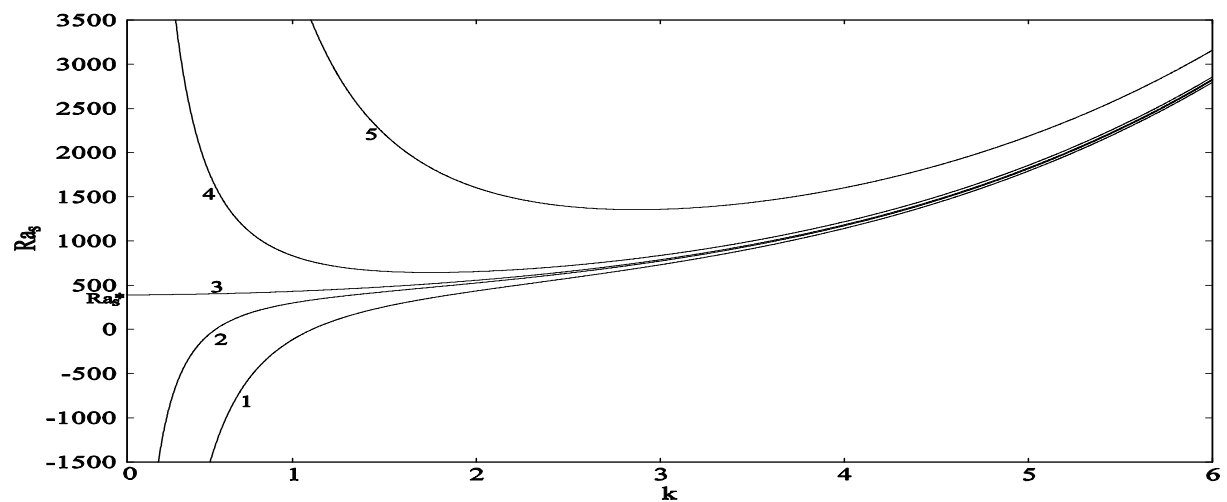


Fig.2: Neutral stability curves for the onset of stationary convection in the case $S=4$ and for several values of Q : (1) $Q=0$; (2) $Q=2$; (3) $Q=2.69680$; (4) $Q=5$ and (5) $Q=30$.

From Fig. 1, we observe that the helical force lowers the threshold of stationary convection when

$Q=10$, while Fig. 2 shows the stabilizing effect of magnetic field when $S=4$.

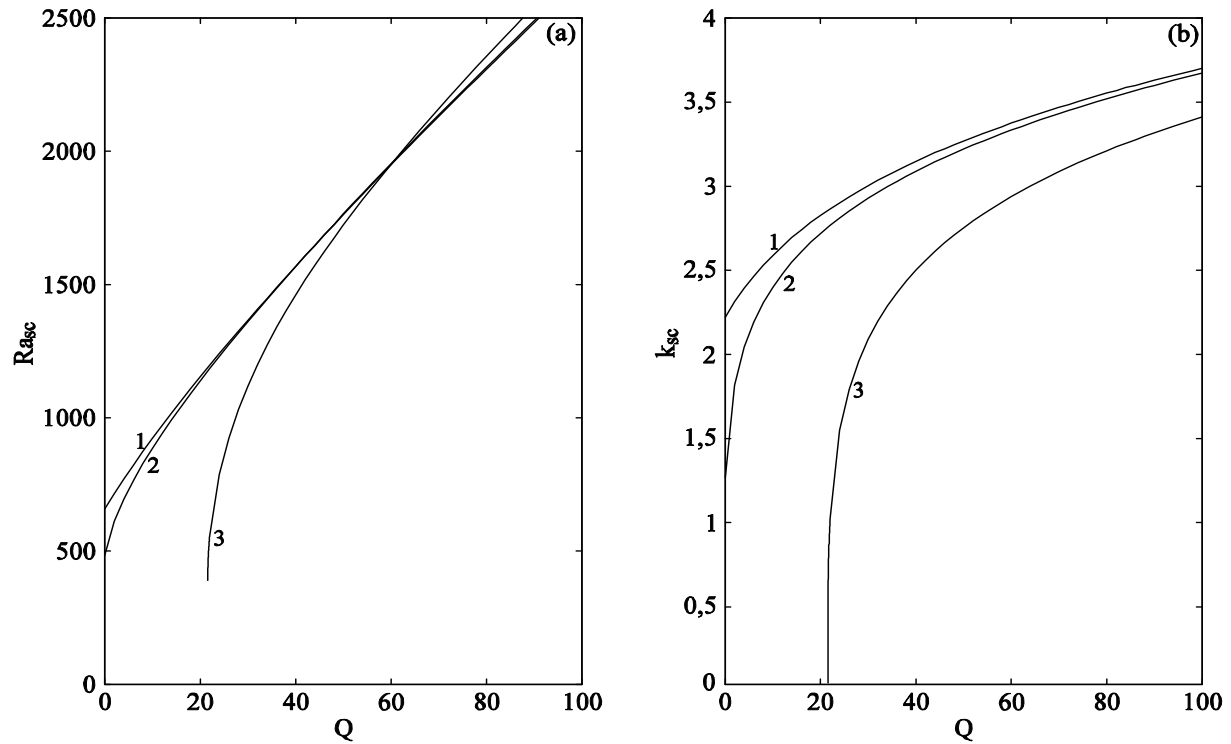


Fig.3: Critical (a) Rayleigh number R_{asc} and (b) wave number k_{sc} as function of Q for several values of S : (1) $S=0$; (2) $S=3$ and (3) $S=10$.

The critical numbers, R_{asc} and k_{sc} as a function of Q for several values of S , are plotted in Fig. 3a and 3b, respectively. We observe that for all Q , the helical force doesn't have the monotonic effect on the onset of stationary convection, but does have

the monotonic decreasing effect on the corresponding critical wave number. Also we can see the horizontal scale of the structures increases when S increases.

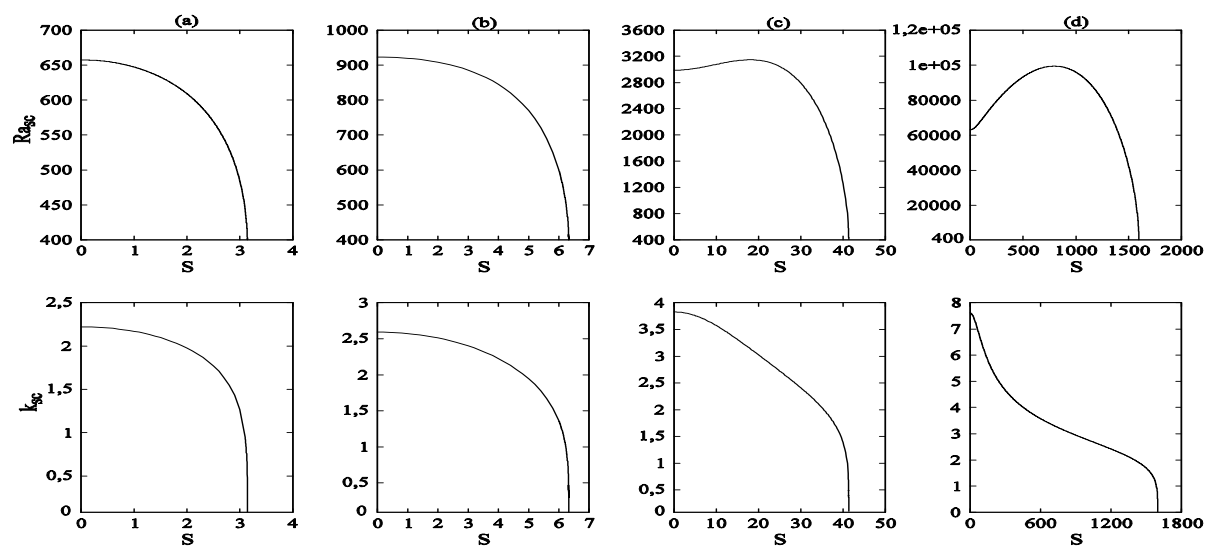


Fig.4: Critical (upper) Rayleigh number R_{asc} and (lower) wave number k_{sc} as function of S for several values of Q : (a) $Q=0$; (b) $Q=10$; (c) $Q=120$ and (d) $Q=5000$. In Fig. 4a, we recover the results obtained by Levina et al. [10].

Numerically, we found ranges for Q ($Q \leq 40$, $40 < Q < 500$ and $Q \geq 500$) in which the effect of the helical force on the threshold of the stationary convection is pointed out. Then, one can conclude

that the influence of the helical force dominates that of the magnetic field when $Q \leq 40$.

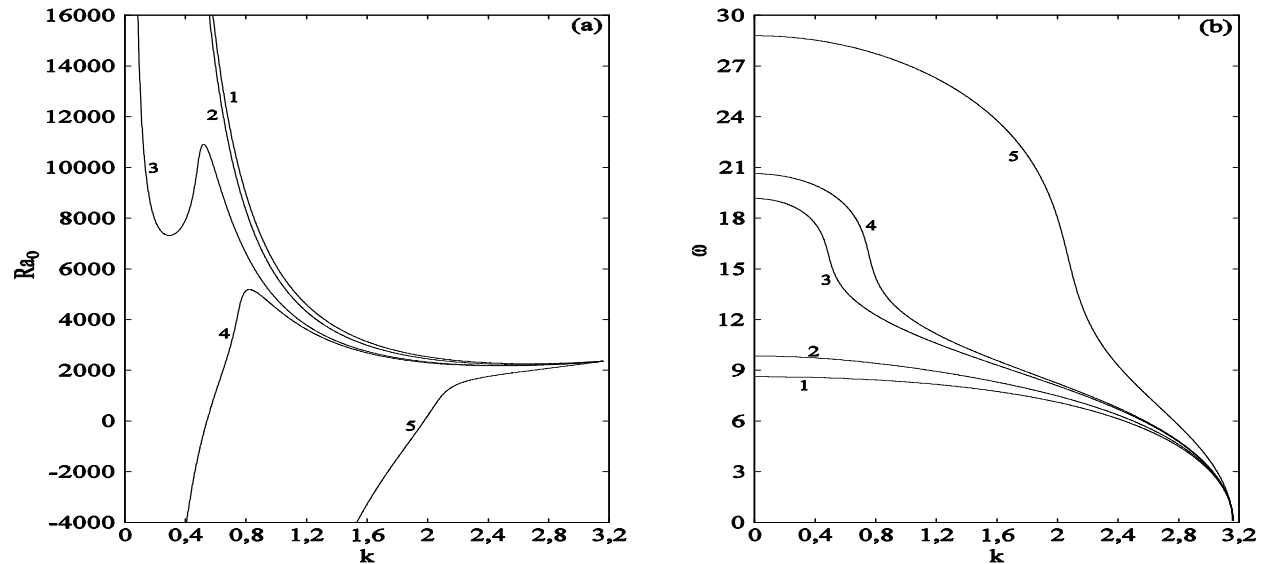


Fig.5: (a) Neutral stability curves for the onset of oscillatory (dashed) convection and (b) frequency ω in the case $Q=80$ and $P_r = 1, P_m = 2$ for several values of S : (1) $S=0$; (2) $S=3$; (3) $S=4.7$; (4) $S=5$; (5) $S=10$.

From Fig. 5 (a) and (b), one notices the helical force effect on the oscillatory convection and the frequency for $Q = 0$.

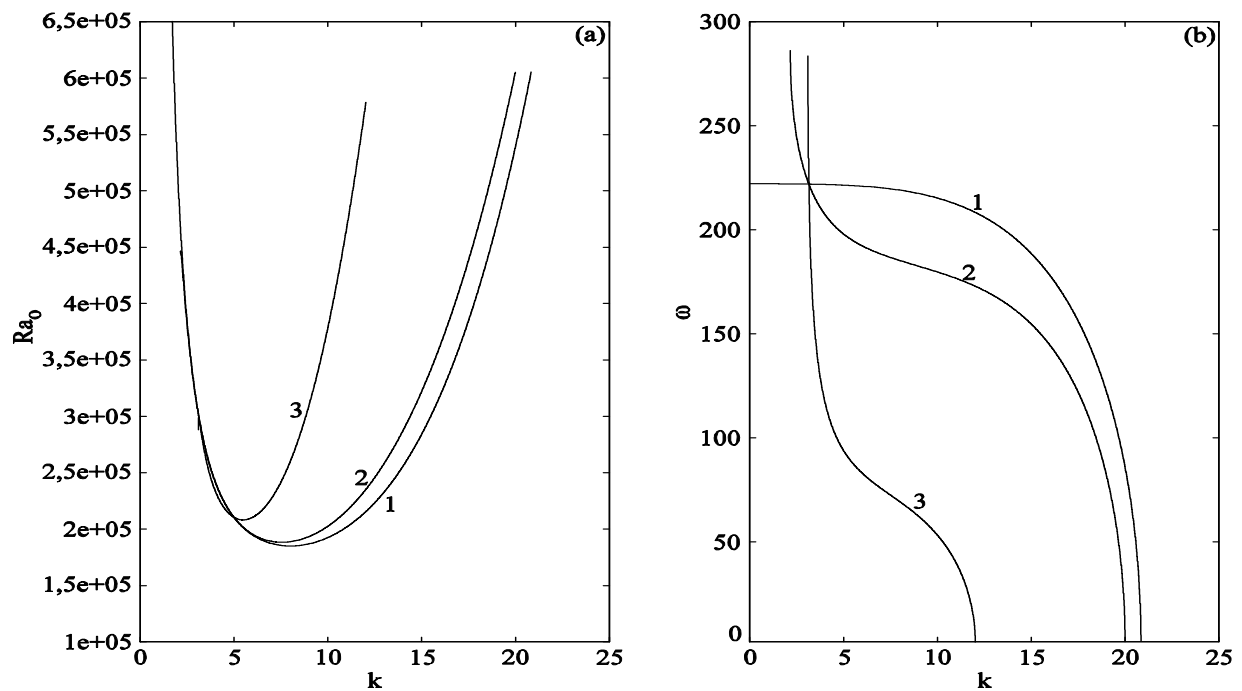


Fig.6: (a) Neutral stability curves for the onset of oscillatory (dashed) convection and (b) frequency ω in the case $Q=40000$ and $P_r = 1, P_m = 2$ for several values of S : (1) $S=0$; (2) $S=100$; (3) $S=500$.

From Fig. 6(a), we found that the helical force increases the threshold of oscillatory convection

and in Fig. 6(b), we observe the changes in the frequency for $Q=40000$.

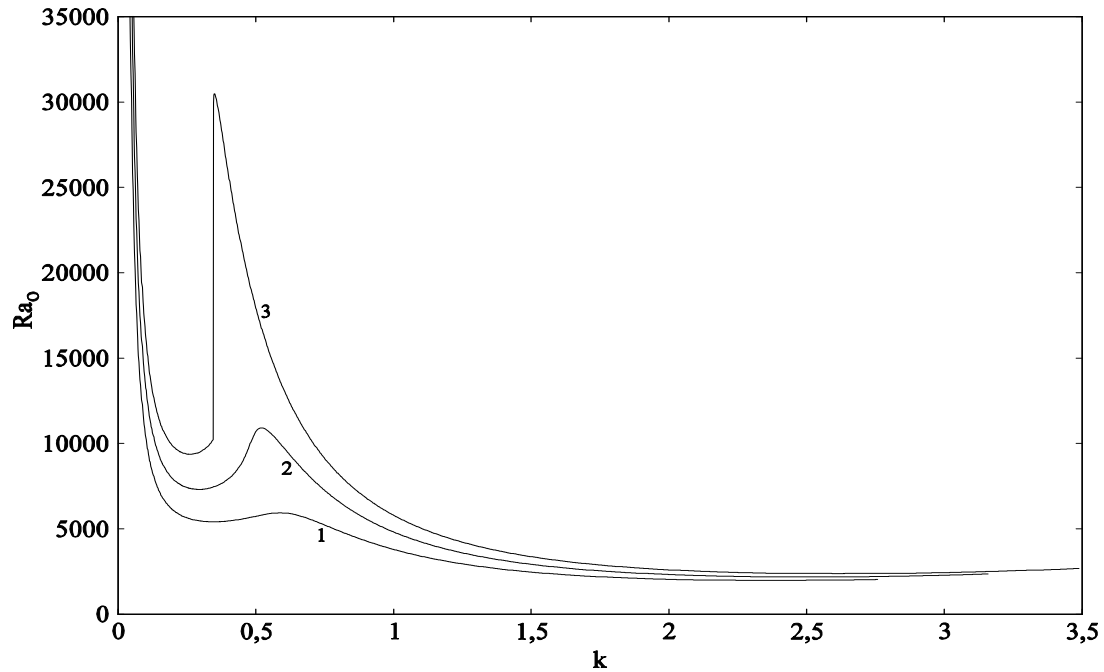


Fig.7: Neutral stability curves for the onset of oscillatory convection in the case $S=4.7$ and $P_r = 1, P_m = 2$ for several values of Q : (1) $Q=60$; (2) $Q=80$; (3) $Q=100$.

From Fig. 7, we observe that for critical values, $S=4.7$, $P_r = 1$ and $P_m = 2$ there exist some values of Q for which curve $Ra_0(k)$ has two minima. In other words, for $40 \leq Q \leq 113.3$, we have the

formation of two cells at marginal stability for that critical value $S=4.7$. Others behaviors, when changing the magnitude of Q , are also plotted.

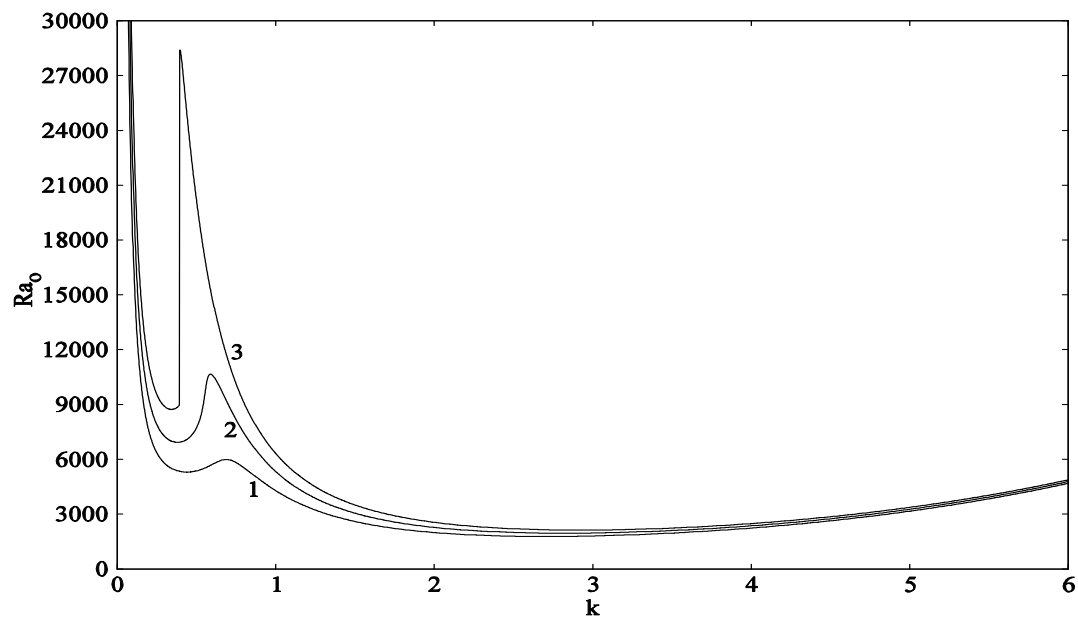


Fig.8: Neutral stability curves for the onset of oscillatory convection in the case $S=3.9$ and $P_r = 1, P_m = 4$ for several values of Q : (1) $Q=200$; (2) $Q=250$; (3) $Q=300$.

In Fig. 8, we notice that the critical value of S , in order to obtain the two minima, i.e., both cells at marginal stability, is lowered and equal to 3.9 for $P_r = 1$ and $P_m = 4$ as compared to the case $P_r = 1$ and $P_m = 2$, where this critical value of S was equal to 4.7. Observing the plots, we can say that both the critical value of S and the smallest value of Q for which we can see the appearance of two cells at marginal stability depend on the magnetic Prandtl number P_m .

As the magnetic field increases still further, the critical Rayleigh number will start decreasing and pass through a minimum; and eventually, the inhibition due to the magnetic field will predominate.

The critical Rayleigh number Ra_{oc} for the onset of oscillatory convection, the critical frequency of oscillation ω_c and the critical wave number k_{oc} , respectively, are plotted versus Q for two values of S as follows.

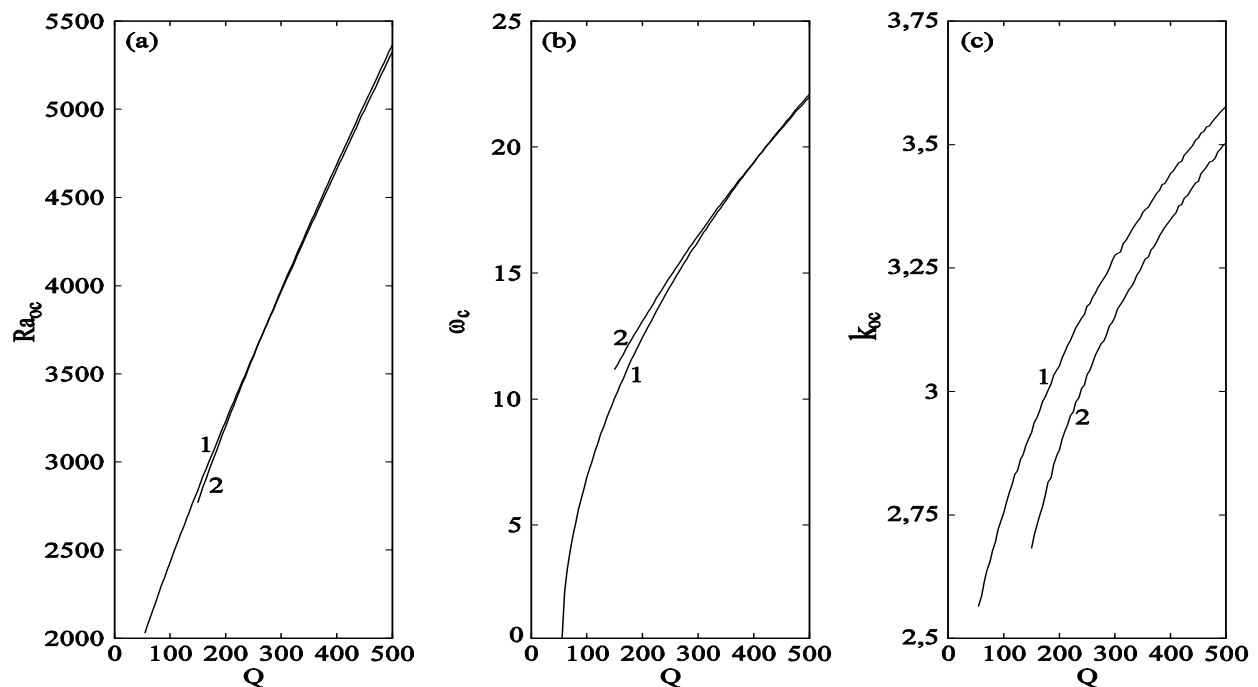


Fig.9: Critical (a) Rayleigh number Ra_{oc} , (b) frequency ω_c and (c) wave number k_{oc} as function of Q for $P_r = 1$ and $P_m = 2$, and for several values of S : (1) $S=0$; (2) $S=7$.

From Fig. 9, we can say that the helical force has similar non-monotonic effects on the onset of oscillatory convection and the monotonic decreasing effect on the corresponding wave number as in the case of stationary convection.

5. Conclusion

In this work, we use the classical linear stability theory to investigate the effects of magnetic field and helical force on the onset of stationary and oscillatory convection in a horizontal electrically conducting fluid layer heated from below with free-free boundary conditions. It has been shown that for all values of Q , the helical force has the non-monotonic effects on both the onset of stationary and oscillatory convections, but has the monotonic

decreasing effect on the corresponding wave number. We observed that the influence of the helical force dominates that of the magnetic field when $Q \leq 40$ in the case of stationary convection. We found the condition for the formation of one large-scale structure. We also have obtained the condition for the formation of two cells at marginal stability in the case of oscillatory convection.

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