

Exact Solutions of the Bianchi Type VI₀ Metric Via Time Dependent Quasi-Maxwell Equations

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The exact solutions of the Einstein field equations for the Bianchi type VI₀ metric via time-dependent quasi-Maxwell equations² when the source of gravitational field is a perfect fluid are obtained.

1. Introduction

A stationary spacetime³ $(M, g_{\mu\nu})$ is a 4-dim Lorentzian manifold with a timelike Killing vector field λ^μ . We consider the case when the observer's velocity components u^μ are in λ^μ direction, where

$$u^\mu = \frac{\lambda^\mu}{\lambda} ; \lambda = \sqrt{g_{\mu\nu}\lambda^\mu\lambda^\nu} \quad (1)$$

In the projection formalism, the metric is decomposed as [1]

$$ds^2 = (u_\sigma dx^\sigma)^2 + (g_{\mu\nu} - u_\mu u_\nu) dx^\mu dx^\nu \quad (2)$$

Next, we choose $\{\lambda^\mu\} = (1, 0, 0, 0)$ and $\{u^\mu\} = (\frac{1}{\sqrt{h}}, 0, 0, 0)$, where h is a function of x^μ , then the metric takes the following form [2-4]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = h(dt - g_i dx^i)^2 - \gamma_{ij} dx^i dx^j \quad (3)$$

where the components of metric are

$$g_{00} = h, \quad g_{0i} = -hg_i, \quad g_{ij} = -\gamma_{ij} + hg_i g_j \quad (4)$$

and their inverse are

$$g^{00} = -g^2 + \frac{1}{h}, \quad g^{0i} = -g^i, \quad g^{ij} = -\gamma^{ij} \quad (5)$$

such that $g^2 = g^i g_i = \gamma^{ij} g_i g_j$. It is interesting to rewrite the Einsteins field equations in terms of gravoelectric and gravomagnetic fields⁴ in γ -space⁵

with time-dependent metric γ_{ij} . Hence, the field equations can be written as the TQME⁶, [7, 8]:

$${}^*\nabla \cdot {}^*\mathbf{E} = {}^*\mathbf{E}^2 + \frac{1}{2} {}^*\mathbf{B}^2 - \frac{{}^*\partial D}{\partial t} - d - \frac{1}{2}(\zeta + U) \quad (6)$$

$${}^*\nabla \times {}^*\mathbf{B} = 2({}^*\mathbf{S} + {}^*\mathbf{M} - \mathbf{j}_m) \quad (7)$$

$$\begin{aligned} {}^*\mathbf{R}_{ij} = & -{}^*\nabla_{(i} {}^*\mathbf{E}_{j)} + {}^*\mathbf{E}_i {}^*\mathbf{E}_j + \frac{1}{2}({}^*\mathbf{B}_i {}^*\mathbf{B}_j - \gamma_{ij} {}^*\mathbf{B}^2) \\ & + 2D_{ik} D_j^k - DD_{ij} + \sqrt{\gamma} \varepsilon_{nk(i} D_{j)}^n {}^*\mathbf{B}^k \\ & - \frac{{}^*\partial D_{ij}}{\partial t} + U_{ij} + \frac{1}{2} \gamma_{ij} (\zeta - U) \end{aligned} \quad (8)$$

Where, $\zeta = \frac{T_{00}}{h}$ is density of the moving substance, $\mathbf{j}_m^i = \frac{T_0^i}{\sqrt{h}}$ is the momentum density, $U_{ij} = T_{ij}$ is 3-dim kinematic stress tensors and $U = U_i^i$ while $T_{\mu\nu}$ are energy-momentum tensors. Also, $\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{h}} \frac{\partial}{\partial t}$, $\gamma = \det(\gamma_{ij})$ and $d = D_{ij} D^{ij}$ such that

$$\begin{aligned} D_{ij} &= \frac{1}{2} \frac{{}^*\partial \gamma_{ij}}{\partial t} \\ D^{ij} &= -\frac{1}{2} \frac{{}^*\partial \gamma^{ij}}{\partial t} \\ D &= \gamma^{ij} D_{ij} = \frac{{}^*\partial \ln \sqrt{\gamma}}{\partial t} \end{aligned} \quad (9)$$

and time-dependent gravoelectric and gravomagnetic fields are defined in terms of gravoelectric

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² Hereinafter abbreviated as TQME.

³ Indices $\alpha, \beta, \dots = 0, 1, 2, 3$ and indices $i, j, \dots = 1, 2, 3$.

⁴ See reference [5] for a discussion of this point.

⁵ The quotient space obtained by quotienting spacetime by the action of the stationary isometry and it represents the collection of orbits of the Killing vectors λ^μ , [6].

⁶ The symbols $()$ and $[]$ represent the commutation and anticommutation over indices, gravitational units with $c=G=1$ are used and the 3-dim Levi-Civita tensor ε_{ijk} is antisymmetric under interchange of any pair of indices such that $\varepsilon_{123} = \varepsilon^{123} = 1$, [3].

potential $\psi = \ln \sqrt{h}$ and gravomagnetic vector potential $\mathbf{g} = (g_1, g_2, g_3)$ is given as follows⁷

$${}^*\mathbf{E} = -{}^*\nabla\psi - \frac{\partial \mathbf{g}}{\partial t}; \quad {}^*\mathbf{E}_i = -\psi_{*i} - \frac{\partial g_i}{\partial t} \quad (10)$$

$$\frac{{}^*\mathbf{B}}{\sqrt{h}} = {}^*\nabla \times \mathbf{g}; \quad \frac{{}^*\mathbf{B}^i}{\sqrt{h}} = \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}} g_{[k*j]} \quad (11)$$

In Eqn. (8), ${}^*\mathbf{R}_{ij}$ is 3-dim starry Ricci tensor constructed from 3-dim starry Christoffel symbols as ${}^*\mathbf{R}_{ij} = {}^*\lambda_{ij*}^k - {}^*\lambda_{ik*}^j + {}^*\lambda_{ij}^n {}^*\lambda_{kn}^k - {}^*\lambda_{in}^k {}^*\lambda_{kj}^n$, where ${}^*\lambda_{jk}^i = \frac{1}{2}\gamma^{il}(\gamma_{jl*}^k + \gamma_{kl*}^j - \gamma_{jk*}^l)$ and also the starry covariant derivatives of an arbitrary 3-vector and a tensor are given by ${}^*\nabla_j \mathbf{A}_i = \mathbf{A}_{i*j} - {}^*\lambda_{ij}^k \mathbf{A}_k$, ${}^*\nabla_j \mathbf{A}^i = \mathbf{A}_{*j}^i + {}^*\lambda_{jk}^i \mathbf{A}^k$ and ${}^*\nabla_k \mathbf{K}^{ij} = \mathbf{K}_{*k}^{ij} + {}^*\lambda_{nk}^i \mathbf{K}^{jn} + {}^*\lambda_{nk}^j \mathbf{K}^{in}$. Finally, the vectors ${}^*\mathbf{S} = {}^*\mathbf{E} \times {}^*\mathbf{B}$ and $\mathbf{M}$ have components as ${}^*S^i = \frac{\varepsilon^{ijk}}{\sqrt{\gamma}} {}^*E_j {}^*B_k$ and ${}^*M^i = -{}^*\nabla_j D^{ij} + {}^*\partial^i D$.

1.1. Exact solutions of the Bianchi type VI₀ metric via TQME

We consider a two complex pole soliton perturbation of a seed metric of the form [9,10]:

$$ds^2 = \xi dt^2 - te^{-2kz} dx^2 - te^{2kz} dy^2 - \xi dz^2 \quad (12)$$

which corresponds to a Bianchi type VI₀ metric, k is a positive constant and ξ is unknown function of t only. A simple calculation shows that all components of gravoelectric and gravomagnetic fields are zero. As before, we need to calculate 3-dim starry Ricci tensors. For doing this, all nonzero components of 3-dim starry Christoffel symbols are calculated as below

$$\begin{aligned} {}^*\lambda_{23}^2 &= -{}^*\lambda_{13}^1 = k \\ {}^*\lambda_{11}^3 &= \frac{kt}{\xi} e^{-2kz} \\ {}^*\lambda_{22}^3 &= -\frac{kt}{\xi} e^{2kz} \end{aligned} \quad (13)$$

Now, these symbols enable us to conclude

$${}^*\mathbf{R}_{ij} = \begin{cases} -2k^2 & i = j = 3, \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

⁷ Note that the divergence and curl of an arbitrary vector in γ -space are defined by ${}^*\nabla \cdot \mathbf{A} = \frac{1}{\sqrt{\gamma}}(\sqrt{\gamma} A^i)_{*i}$ and

$$({}^*\nabla \times \mathbf{A})^i = \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}} A_{[k*j]} \text{ while } {}^*_{*i} = {}^*\partial_i = \partial_i + g_i \frac{\partial}{\partial t}.$$

On the other hand, we assume that the matter behaves as a perfect fluid, i.e.,

$$\mathbf{T}_{\mu\nu} = (\rho + p)w_\mu w_\nu - pg_{\mu\nu} \quad (15)$$

together with $w^\alpha w_\alpha = 1$ and $w_\alpha = (1, 0, 0, 0)$. Here ρ , p and u_μ are, respectively, the matter density, pressure and 4-velocity vector of the matter distribution. In continuation, a simple calculation leads us to

$$\mathbf{U} = 3p \quad (16)$$

$$\mathbf{D} = \frac{at + 2}{2t\sqrt{\xi}} \quad (17)$$

Where⁸ $a = \frac{\dot{\xi}}{\xi}$. Substitution of Eqns. (14)-(17) into Eqn. (8) lead to the following equations

$$\rho = (2\xi - 1)p \quad (18)$$

$$\dot{a} + \frac{a}{t} - \rho + (2\xi - 1)p - 4k^2 = 0 \quad (19)$$

Comparing these equations one finds

$$\dot{a} + \frac{a}{t} - 4k^2 = 0 \quad (20)$$

By solving this equation, we conclude that

$$\xi = t^n e^{k^2 t^2} \quad (21)$$

Where, n is a constant. To continue our analysis, we need the law of energy equations [8] given as follows

$${}^*\nabla \cdot \mathbf{j}_m + \frac{{}^*\partial \zeta}{\partial t} + \mathbf{D}\zeta + \mathbf{U}_{ij} \mathbf{D}^{ij} - 2\mathbf{j}_m^k {}^*\mathbf{E}_k = 0 \quad (22)$$

$$\frac{{}^*\partial \mathbf{j}_m}{\partial t} + \mathbf{D}\mathbf{j}_m - \zeta {}^*\mathbf{E} - \mathbf{j}_m \times {}^*\mathbf{B} + \mathbf{\Xi} = 0 \quad (23)$$

here $\mathbf{\Xi}^i = {}^*\nabla_k \mathbf{U}^{ik} - {}^*\mathbf{E}_k \mathbf{U}^{ik}$. Using Eqn. (18), we see that $\zeta = p$ and $\mathbf{U}_{ij} \mathbf{D}^{ij} = \mathbf{D}p$. By putting these results and Eqn. (21) into Eqn. (22), and after simplification yields

$$p = \frac{c}{\xi t^2} \quad (24)$$

Where, c is a real constant. From Eqn. (6), together with the last consequence, we get

$$c = \frac{n+2}{4} \quad (25)$$

⁸ Here, dot stand for differentiation with respect to t .

Therefore, from Eqns. (18) and (24), we deduce

$$\rho = \frac{(n+2)(2\xi-1)}{4\xi t^2} \quad (26)$$

Furthermore, it is easy to prove that

$${}^*\mathbf{M} = \mathbf{\Xi} = 0 \quad (27)$$

Lastly, with the help of Eqn. (27), we cannot get any equation from Eqns. (7) and (23).

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