

Dirac Equation with Generalized Yukawa Potential and Tensor Terms

Akpan N. Ikot^{*1}, Elham Maghsoodi², Eno J. Ibanga¹, Saber Zarrinkamar³ and Hassan Hassanabadi⁴

¹Theoretical Physics Group, Department of Physics, University of Uyo, Nigeria

²Department of Basic Sciences, Shahrood Branch, Islamic Azad University, Shahrood, Iran

³Department of Basic Sciences, Garmsar Branch, Islamic Azad University, Garmsar, Iran

⁴Department of Physics, Shahrood University of Technology, Shahrood, Iran

We solve the spin and pseudospin symmetries of Dirac equation with a generalized Yukawa potential and a Yukawa tensor term in an approximate analytical manner. The eigenfunctions and the energy eigenvalues are reported via the Nikiforov-Uvarov technique and discuss the problem in terms of the various parameters engaged.

1. Introduction

The study of bound-state solutions of the Schrödinger, Klein-Gordon and Dirac equations for spherically symmetric potentials plays a vital role in various fields of physics. The bound-state solutions of such potentials can effectively be used to model some physical problems. The Dirac equation is a well-known relativistic wave equation describing spin- $\frac{1}{2}$ particles [1]. However, the analytical solutions of the Dirac equation are possible only for a few simple cases, such as the hydrogen atom and the harmonic oscillator [2-3]. In recent years, many researchers have reported various solutions of the Dirac equation with spherically symmetric potentials [4]. These potentials include Pöschl-Teller [5], Hulthén [6], Pseudo-harmonic [7], Hylleraas [8], Kratzer-Fues [9], Yukawa [10] and some others [11]. In these studies, various methods have been used to solve these quantum mechanical problems in exact or approximate analytical manners. These methods include the Nikiforov-Uvarov (NU) method [12], asymptotic iteration method (AIM) [13], supersymmetric quantum mechanics (SUSYQM), [14] and others [15]. Among a rather lengthy list of interactions, the Yukawa potential stands as an appealing choice for particle physicists. Related works on the Dirac equation and the Yukawa potential can be found in [17,18]. The Yukawa potential is one of the most studied potential in physics as it has been used extensively in different area such as solid state physics and atomic physics. On the other hand, we are now almost sure that

the so-called spin and pseudo-spin symmetries of Dirac equation have significant role in nuclear and hadronic spectroscopy [18]. In this formulation, the tensor interaction has attracted a great attention as it removes the degeneracy between the doublets [16]. In most of these studies, due to the mathematical structure of this problem, the tensor interaction is considered as Coulomb-like [19] or Cornell interaction. Hassanabadi et al. were the first who introduces the Yukawa tensor interaction [16,20]. Here, we consider the generalized Yukawa potential besides a tensor Yukawa term and proceed on an approximate analytical manner. In the next section, we will introduce the Nikiforov-Uvarov (NU) technique and will use it to solve the differential equation obtained.

2. The Nikiforov-Uvarov Method

The NU method can solve a second-order differential equation of the form [12]

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0 \quad (1)$$

Where, $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and more direct, we introduce a more compact presentation of the idea. In order to do this, we rewrite Eqn. (1) as follows [21]

^{*}ndemikotphysics@gmail.com

$$\begin{aligned} \psi_n''(s) + \left(\frac{c_1 - c_2 s}{s(1 - c_3 s)} \right) \psi_n'(s) \\ + \left(\frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{s_2(1 - c_3 s)^2} \right) \psi_n(s) = 0 \end{aligned} \quad (2)$$

in which

$$\psi_n(s) = \phi(s)y_n(s) \quad (3)$$

Comparing Eqn. (1) with Eqn. (2), we obtain the following identifications

$$\tilde{\tau}(s) = c_1 - c_2 s, \quad \sigma(s) = s(1 - c_3 s)$$

$$\tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3 \quad (4)$$

Following the NU method [12,21], we obtain the following required parameters:

(i) The relevant constant:

$$c_4 = \frac{1}{2}(1 - c_1), \quad c_5 = \frac{1}{2}(c_2 - 2c_3),$$

$$c_6 = c_5^2 + \xi_1, \quad c_7 = 2c_4 c_5 - \xi_2,$$

$$c_8 = c_4^2 + \xi_3, \quad c_9 = c_3 c_7 + c_3^2 c_8 + c_6,$$

$$c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}$$

$$c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8})$$

$$c_{12} = c_4 + \sqrt{c_8} \quad c_{13} = c_5 - (\sqrt{c_9} + c_3\sqrt{c_8}) \quad (5)$$

(ii) The essential polynomial functions:

$$\pi(s) = c_4 + c_5 s - \left[(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8} \right] \quad (6)$$

$$k = -(c_7 + 2c_3 c_8) - 2\sqrt{c_8 c_9} \quad (7)$$

$$\begin{aligned} \tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)s \\ - 2 \left[(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8} \right] \end{aligned} \quad (8)$$

$$\tau'(s) = -2c_3 - 2(\sqrt{c_9} + c_3\sqrt{c_8}) < 0 \quad (9)$$

(iii) The energy equation:

$$\begin{aligned} c_2 n - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) \\ + n(n-1)c_3 + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0 \end{aligned} \quad (10)$$

(iv) The wave functions

$$\rho(s) = s^{c_{10}} (1 - c_3 s)^{c_{11}} \quad (11)$$

$$\phi(s) = s^{c_{12}} (1 - c_3 s)^{c_{13}}, \quad c_{12} > 0, c_{13} > 0 \quad (12)$$

$$y_n(s) = P_n^{(c_{10}, c_{11})}(1 - 2c_3 s), c_{10} > -1, c_{11} > -1 \quad (13)$$

$$\psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}} (1 - c_3 s)^{-c_{12} \frac{c_{13}}{c_3}} P_n^{(c_{10}-1, \frac{c_{11}}{c_3}-c_{10}-1)}(1 - 2c_3 s) \quad (14)$$

Where, $P_n^{(\mu, \nu)}(x)$, $\mu > -1, \nu > -1$ and $x \in [-1, 1]$ are Jacobi polynomials with

$$P_n^{(\alpha, \beta)}(1 - 2s) = \frac{(\alpha+1)_n}{n!} {}_2F_1(-n, 1 + \alpha + \beta + n; \alpha + 1; s) \quad (15)$$

and $N_{n\kappa}$ is a normalization constant. Also, the above wave functions can be expressed in terms of the hyper-geometric function via

$$\begin{aligned} \psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}} (1 - c_3 s)^{c_{13}} {}_2F_1(-n, 1 + c_{10} \\ + c_{11} + n; c_{10} + 1; c_3 s) \end{aligned} \quad (16)$$

Where, $c_{12} > 0, c_{13} > 0$ and $s \in [0, 1/c_3], c_3 \neq 0$. This method has been extensively used to solve various second-order differential equations in quantum mechanics such as Schrödinger, Klein-Gordon, Duffin-Kemmer-Petiau (DKP), spinless-Salpeter, and Dirac equations [22].

3. Theory of Dirac Equation

The Dirac equation for spin- $\frac{1}{2}$ particles moving in an attractive scalar potential $S(r)$, a repulsive vector potential $V(r)$ and a tensor potential $U(r)$ in the relativistic unit ($\hbar = c = 1$) [23] is

$$[\vec{\alpha} \cdot \vec{p} + \beta(M + S(r)) - i\beta\vec{\alpha} \cdot \hat{r}U(r)]\psi(r) = [E - V(r)]\psi(r) \quad (17)$$

Where, E is the relativistic energy of the system, $\vec{p} = -i\vec{\nabla}$ is the three dimensional momentum operator and M is the mass of the fermionic particle. $\vec{\alpha}, \beta$, are the 4×4 Dirac matrices given as

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma}_i \\ \vec{\sigma}_i & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (18)$$

Where, I is 2×2 unitary matrix and $\vec{\sigma}_i$ are the Pauli three-vector matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (19)$$

The eigen-values of the spin-orbit coupling operator are $\kappa = \left(j + \frac{1}{2}\right) > 0, \kappa = -\left(j + \frac{1}{2}\right) < 0$ respectively for unaligned $j = l - \frac{1}{2}$ and aligned spin $j = l + \frac{1}{2}$. The set (H, K, J^2, J_z) forms a complete set of conserved quantities. Thus, we can write the spinors [24] as

$$\psi_{n\kappa}(r) = \frac{1}{r} \begin{pmatrix} F_{n\kappa}(r)Y_{jm}^l(\theta, \varphi) \\ iG_{n\kappa}(r)Y_{jm}^l(\theta, \varphi) \end{pmatrix} \quad (20)$$

Where, $F_{n\kappa}(r), G_{n\kappa}(r)$ represent the upper and lower components of the Dirac spinors. $Y_{jm}^l(\theta, \varphi)$, $Y_{jm}^l(\theta, \varphi)$ are the spin and pseudo-spin spherical harmonics and m is the projection on the z-axis. Using well-known identities [24]

$$(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\vec{\sigma} \cdot (\vec{A} \times \vec{B}),$$

$$\vec{\sigma} \cdot \vec{p} = \vec{\sigma} \cdot \hat{r} \left(\hat{r} \cdot \vec{p} + i \frac{\vec{\sigma} \cdot \vec{L}}{r} \right) \quad (21)$$

as well as the relations

$$(\vec{\sigma} \cdot \vec{L})Y_{jm}^l(\theta, \varphi) = (\kappa - 1)Y_{jm}^l(\theta, \varphi)$$

$$(\vec{\sigma} \cdot \vec{L})Y_{jm}^l(\theta, \varphi) = -(\kappa - 1)Y_{jm}^l(\theta, \varphi)$$

$$(\vec{\sigma} \cdot \hat{r})Y_{jm}^l(\theta, \varphi) = -Y_{jm}^l(\theta, \varphi)$$

$$(\vec{\sigma} \cdot \hat{r})Y_{jm}^l(\theta, \varphi) = Y_{jm}^l(\theta, \varphi) \quad (22)$$

we find the following two coupled first-order Dirac equation [24]

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{n\kappa}(r) = (M + E_{n\kappa} - \Delta(r)) G_{n\kappa}(r) \quad (23)$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{n\kappa}(r) = (M - E_{n\kappa} + \Sigma(r)) F_{n\kappa}(r) \quad (24)$$

where,

$$\Delta(r) = V(r) - S(r) \quad (25)$$

$$\Sigma(r) = V(r) + S(r) \quad (26)$$

Eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ in Eqns. (23) and (24), we obtain the second-order Schrödinger-like equation

$$\left\{ \begin{aligned} & \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa U(r)}{r} - \frac{dU(r)}{dr} - U^2(r) \\ & - (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) \\ & + \frac{\frac{d\Delta(r)}{dr} \left(\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right)}{(M + E_{n\kappa} - \Delta(r))} \end{aligned} \right\} F_{n\kappa}(r) = 0 \quad (27)$$

$$\left\{ \begin{aligned} & \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa U(r)}{r} + \frac{dU(r)}{dr} - U^2(r) \\ & - (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) \\ & + \frac{\frac{d\Sigma(r)}{dr} \left(\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right)}{(M + E_{n\kappa} - \Sigma(r))} \end{aligned} \right\} G_{n\kappa}(r) = 0 \quad (28)$$

Where, $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$, $\kappa(\kappa+1) = l(l+1)$.

3.1. Pseudo-spin symmetry limit

In the pseudo-spin symmetry limit, $\frac{d\Sigma(r)}{dr} = 0$, or $\Sigma(r) = C_{ps} = \text{const}$ [18]. Here, we consider the generalized Yukawa potential

$$\Delta(r) = -V_0 \left(A + \frac{Be^{-\alpha r}}{r} \right)^2 \quad (29)$$

besides the Yukawa tensor interaction [16].

$$U(r) = -V_1 \left(\frac{e^{-\alpha r}}{r} \right) \quad (30)$$

Substituting Eqns.(29) and (30) into Eqn. (28) yields,

$$\left\{ \begin{aligned} & \frac{d^2}{dr^2} + \frac{V_1 e^{-\alpha r} - V_1^2 e^{-2\alpha r} - 2V_1 \kappa e^{-\alpha r} - \kappa(\kappa-1)}{r^2} \\ & + \frac{\alpha V_1 e^{-\alpha r}}{r} - \varepsilon^2 V_0 \gamma \left(A^2 + \frac{2ABe^{-\alpha r}}{r} + \frac{B^2 e^{-2\alpha r}}{r^2} \right) \end{aligned} \right\} G_{n\kappa}(r) = 0 \quad (31)$$

Where,

$$\varepsilon^2 = (M + E_{n\kappa})\gamma, \gamma = (M - E_{n\kappa} + C_{ps}) \quad (32)$$

Introducing the approximations [16,25] (see Fig. 1)

$$\begin{aligned} \frac{1}{r^2} &\approx \frac{4\alpha^2 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \\ \frac{1}{r} &\approx \frac{4\alpha^2 e^{-\alpha r}}{(1-e^{-2\alpha r})^2} \end{aligned} \quad (33)$$

Eqn. (31) appears as

$$\left\{ \begin{aligned} & \frac{d^2}{dr^2} - \frac{4\alpha^2 \kappa(\kappa-1)e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} - \frac{8\alpha^2 V_1 \kappa e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \\ & + \frac{2\alpha^2 V_1 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} + \frac{4\alpha^2 V_1 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \\ & - \frac{4\alpha^2 V_1^2 e^{-4\alpha r}}{(1-e^{-2\alpha r})^2} - \varepsilon^2 \\ & - V_0 \gamma \left(A^2 + \frac{4\alpha A B e^{-2\alpha r}}{(1-e^{-2\alpha r})} + \frac{4\alpha^2 B^2 e^{-4\alpha r}}{(1-e^{-2\alpha r})^2} \right) \end{aligned} \right\} G_{n\kappa}(r) = 0 \quad (34)$$

By applying the transformation, $s = e^{-2\alpha r}$, Eqn. (34) takes the form

$$\begin{aligned} & \frac{d^2 G_{n\kappa}(s)}{ds^2} + \frac{(1-s)}{s(1-s)} \frac{dG_{n\kappa}(s)}{ds} \\ & + \frac{1}{s^2(1-s)^2} (-Q_1^{ps} s^2 + Q_2^{ps} s - Q_3^{ps}) G_{n\kappa}(s) = 0 \end{aligned} \quad (35)$$

Where,

$$\begin{aligned} Q_1^{ps} &= \frac{V_1}{2} + V_1^2 + \frac{M^2}{4\alpha^2} + \frac{MC_{ps}}{4\alpha^2} - \frac{(E_{n\kappa}^{ps})^2}{4\alpha^2} + \frac{E_{n\kappa}^{ps} C_{ps}}{4\alpha^2} \\ & + \frac{MV_0 A^2}{4\alpha^2} - \frac{E_{n\kappa}^{ps} V_0 A^2}{4\alpha^2} + \frac{C_{ps} V_0 A^2}{4\alpha^2} + MV_0 B^2 - E_{n\kappa}^{ps} V_0 B^2 \\ & + C_{ps} V_0 B^2 - \frac{V_0 MAB}{\alpha} + \frac{V_0 E_{n\kappa}^{ps} AB}{\alpha} - \frac{V_0 C_{ps} AB}{\alpha} \end{aligned}$$

$$\begin{aligned} Q_2^{ps} &= -\kappa(\kappa-1) - 2V_1 \kappa + \frac{V_1}{2} + V_1^2 + \frac{M^2}{2\alpha^2} + \frac{MC_{ps}}{2\alpha^2} \\ & - \frac{(E_{n\kappa}^{ps})^2}{2\alpha^2} + \frac{E_{n\kappa}^{ps} C_{ps}}{2\alpha^2} + \frac{MV_0 A^2}{2\alpha^2} - \frac{E_{n\kappa}^{ps} V_0 A^2}{2\alpha^2} + \frac{C_{ps} V_0 A^2}{2\alpha^2} \\ & - \frac{MV_0 AB}{\alpha} + \frac{V_0 ABE_{n\kappa}^{ps}}{\alpha} - \frac{V_0 ABC_{ps}}{\alpha} \end{aligned}$$

$$\begin{aligned} Q_3^{ps} &= \frac{M^2}{4\alpha^2} - \frac{(E_{n\kappa}^{ps})^2}{4\alpha^2} + \frac{MC_{ps}}{4\alpha^2} + \frac{E_{n\kappa}^{ps} C_{ps}}{4\alpha^2} + \frac{MV_0 A^2}{4\alpha^2} \\ & - \frac{E_{n\kappa}^{ps} V_0 A^2}{4\alpha^2} + \frac{C_{ps} V_0 A^2}{4\alpha^2} \end{aligned} \quad (36)$$

Comparing Eqn. (35) with Eqn. (2), we obtain the following coefficients

$$c_1 = 1, c_2 = 1, c_3 = 1, \xi_1 = Q_1^{ps}, \xi_2 = Q_2^{ps}, \xi_3 = Q_3^{ps} \quad (37)$$

And the other coefficients are determined from Eqns. (5) as

$$\begin{aligned} c_4 &= 0, c_5 = -\frac{1}{2}, c_6 = \frac{1}{4} + Q_1^{ps}, c_7 = -Q_2^{ps}, c_8 = -Q_3^{ps}, \\ c_9 &= \frac{1}{4} + Q_1^{ps} + Q_3^{ps} - Q_2^{ps}, c_{10} = 1 + 2\sqrt{Q_3^{ps}}, \\ c_{11} &= 2 + 2\left(\sqrt{\frac{1}{4} + Q_1^{ps} + Q_3^{ps} - Q_2^{ps}} + \sqrt{Q_3^{ps}}\right), \\ c_{12} &= \sqrt{Q_3^{ps}}, \\ c_{13} &= -\frac{1}{2} - \left(\sqrt{\frac{1}{4} + Q_1^{ps} + Q_3^{ps} - Q_2^{ps}} + \sqrt{Q_3^{ps}}\right) \end{aligned} \quad (38)$$

The energy level for the pseudospin symmetry limit is obtained from Eqn. (10) as

$$\begin{aligned} &n + \left(\frac{2n+1}{2}\right) + (2n+1)\left(\sqrt{\frac{1}{4} + Q_1^{ps} + Q_3^{ps} - Q_2^{ps}} + \sqrt{Q_3^{ps}}\right) \\ &+ n(n-1) - Q_2^{ps} + 2Q_3^{ps} \\ &+ 2\sqrt{Q_3^{ps}}\left(\frac{1}{4} + Q_1^{ps} + Q_3^{ps} - Q_2^{ps}\right) = 0 \end{aligned} \quad (39)$$

Eqn. (39) can be alternatively written as

$$\begin{aligned} &n^2 + n + \frac{1}{2} + \kappa(\kappa-1) + 2V_1\kappa - \frac{V_1}{2} - V_1 + \frac{V_0AB\gamma}{\alpha} \\ &+ (2n+1)\left(\sqrt{\frac{1}{4} + V_1^2 + V_0B^2\gamma + \kappa(\kappa-1) + 2V_1\kappa - V_1} + \sqrt{\frac{M^2}{4\alpha^2} - \frac{(E_{n\kappa}^{ps})^2}{4\alpha^2} + \frac{MC_{ps}}{4\alpha^2} + \frac{E_{n\kappa}^{ps}C_{ps}}{4\alpha^2} + \frac{V_0A^2\gamma}{4\alpha^2}}\right) \\ &+ 2\sqrt{\left(\frac{M^2}{4\alpha^2} - \frac{(E_{n\kappa}^{ps})^2}{4\alpha^2} + \frac{MC_{ps}}{4\alpha^2} + \frac{E_{n\kappa}^{ps}C_{ps}}{4\alpha^2} + \frac{V_0A^2\gamma}{4\alpha^2}\right)\left(\frac{1}{4} + V_1^2 + V_0B^2\gamma + \kappa(\kappa-1) + 2V_1\kappa - V_1\right)} = 0 \end{aligned} \quad (40)$$

and the corresponding components of the lower and upper wave functions are obtained as follows

$$G_{n\kappa}^{ps}(r) = N_{n\kappa}(e^{-2\alpha r})^{\sqrt{Q_3^{ps}}} (1 - e^{-2\alpha r})^{\frac{1}{2} + \sqrt{\frac{1}{4} + Q_1^{ps} + Q_3^{ps} - Q_2^{ps}}} P_n\left(2\sqrt{Q_3^{ps}}, 2\sqrt{\frac{1}{4} + Q_1^{ps} + Q_3^{ps} - Q_2^{ps}}\right) (1 - 2e^{-2\alpha r}) \quad (41)$$

$$F_{n\kappa}^{ps}(r) = \frac{1}{M - E_{n\kappa}^{ps} + C_{ps}} \left(\frac{d}{dr} - \frac{\kappa}{r} - V_1 \left(\frac{e^{-\alpha r}}{r} \right) \right) G_{n\kappa}^{ps}(r) \quad (42)$$

Where, $E_{n\kappa} \neq M + C_{ps}$.

3.2. Spin symmetry limit

In the spin symmetry limit, $\frac{d\Delta(r)}{dr} = 0$, or $\Delta(r) = C_s = \text{const}$ [18]. Substitution of the Generalized Yukawa potential

$$\Sigma(r) = -V_0 \left(A + \frac{Be^{-\alpha r}}{r} \right) \quad (43)$$

into Eqn. (27) yields

$$\begin{aligned} &\left\{ \frac{d^2}{dr^2} + \frac{-V_1e^{-\alpha r} - V_1^2e^{-2\alpha r} - 2V_1\kappa e^{-\alpha r} - \kappa(k+1)}{r^2} \right. \\ &\left. - \frac{\alpha V_1e^{-\alpha r}}{r} + \mu^2 - V_0\delta \left(A^2 + \frac{2ABe^{-\alpha r}}{r} + \frac{B^2e^{-2\alpha r}}{r^2} \right) \right\} F_{n\kappa}^s(r) = 0 \end{aligned} \quad (44)$$

Where,

$$\mu^2 = (M - E_{n\kappa}^s)\delta, \quad \delta = (-M - E_{n\kappa}^s + C_s) \quad (45)$$

Applying the approximation of Eqn. (44) appears as

$$\left\{ \begin{aligned} & \frac{d^2}{dr^2} - \frac{4\alpha^2 \kappa(\kappa+1)e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} - \frac{8\alpha^2 V_1 \kappa e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \\ & - \frac{2\alpha^2 V_1 e^{-2\alpha r}}{(1-e^{-2\alpha r})} - \frac{4\alpha^2 V_1 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \\ & - \frac{4\alpha^2 V_1^2 e^{-4\alpha r}}{(1-e^{-2\alpha r})^2} + \mu^2 \\ & - V_0 \delta \left(A^2 + \frac{4\alpha A B e^{-2\alpha r}}{(1-e^{-2\alpha r})} + \frac{4\alpha^2 B^2 e^{-4\alpha r}}{(1-e^{-2\alpha r})^2} \right) \end{aligned} \right\} F_{n\kappa}^s(r) = 0 \quad (46)$$

By considering $s = e^{-2\alpha r}$, Eqn. (46) comes into the form

$$\begin{aligned} & \frac{d^2 F_{n\kappa}^s(s)}{ds^2} + \frac{(1-s)}{s(1-s)} \frac{dF_{n\kappa}^s(s)}{ds} \\ & + \frac{1}{s^2(1-s)^2} (-Q_1^s s^2 + Q_2^s s - Q_3^s) F_{n\kappa}^s(s) = 0 \quad (47) \end{aligned}$$

Where,

$$\begin{aligned} Q_1^s &= -\frac{V_1}{2} + V_1^2 + \frac{M^2}{4\alpha^2} - \frac{MV_0 A^2}{4\alpha^2} - MV_0 B^2 + \frac{V_0 MAB}{\alpha} \\ & - \frac{(E_{n\kappa}^s)^2}{4\alpha^2} - \frac{E_{n\kappa}^s V_0 A^2}{4\alpha^2} - E_{n\kappa}^s V_0 B^2 + \frac{V_0 E_{n\kappa}^s AB}{\alpha} - \frac{MC_s}{4\alpha^2} \\ & + \frac{E_{n\kappa}^s C_s}{4\alpha^2} + \frac{C_s V_0 A^2}{4\alpha^2} + C_s V_0 B^2 - \frac{V_0 C_s AB}{\alpha} \\ Q_2^s &= -\kappa(\kappa+1) - 2V_1 \kappa - \frac{V_1}{2} + V_1 + \frac{M^2}{2\alpha^2} - \frac{MV_0 A^2}{2\alpha^2} \\ & + \frac{MV_0 AB}{\alpha} - \frac{(E_{n\kappa}^s)^2}{2\alpha^2} - \frac{E_{n\kappa}^s C_{ps}}{2\alpha^2} - \frac{E_{n\kappa}^s V_0 A^2}{2\alpha^2} + \frac{V_0 ABE_{n\kappa}^s}{\alpha} \\ & - \frac{MC_s}{2\alpha^2} + \frac{E_{n\kappa}^s C_s}{2\alpha^2} + \frac{C_s V_0 A^2}{2\alpha^2} - \frac{V_0 ABC_s}{\alpha} \end{aligned}$$

$$\begin{aligned} Q_3^s &= \frac{M^2}{4\alpha^2} - \frac{MV_0 A^2}{4\alpha^2} - \frac{(E_{n\kappa}^s)^2}{4\alpha^2} - \frac{E_{n\kappa}^s V_0 A^2}{4\alpha^2} - \frac{MC_s}{4\alpha^2} \\ & + \frac{E_{n\kappa}^s C_s}{4\alpha^2} + \frac{C_s V_0 A^2}{4\alpha^2} \end{aligned} \quad (48)$$

Comparing Eqn. (48) with Eqn. (2), we obtain the following coefficients

$$c_1 = 1, c_2 = 1, c_3 = 1, \xi_1 = Q_1^s, \xi_2 = Q_2^s, \xi_3 = Q_3^s \quad (49)$$

and

$$c_4 = 0, c_5 = -\frac{1}{2}, c_6 = \frac{1}{4} + Q_1^s, c_7 = -Q_2^s, c_8 = -Q_3^s,$$

$$c_9 = \frac{1}{4} + Q_1^s + Q_3^s - Q_2^s, c_{10} = 1 + 2\sqrt{Q_3^s},$$

$$c_{11} = 2 + 2\left(\sqrt{\frac{1}{4} + Q_1^s + Q_3^s - Q_2^s} + \sqrt{Q_3^s}\right), \quad (50)$$

$$c_{12} = \sqrt{Q_3^s},$$

$$c_{13} = -\frac{1}{2} - \left(\sqrt{\frac{1}{4} + Q_1^s + Q_3^s - Q_2^s} + \sqrt{Q_3^s}\right)$$

And the energy relation for the spin symmetry limit, which is

$$\begin{aligned} & n + \left(\frac{2n+1}{2}\right) + (2n+1) \left(\sqrt{\frac{1}{4} + Q_1^s + Q_3^s - Q_2^s} + \sqrt{Q_3^s}\right) \\ & + n(n-1) - Q_2^s + 2Q_3^s \\ & + 2\sqrt{Q_3^s} \left(\frac{1}{4} + Q_1^s + Q_3^s - Q_2^s\right) = 0 \end{aligned} \quad (51)$$

or

$$\begin{aligned}
& n^2 + n + \frac{1}{2} + \kappa(\kappa+1) + 2V_1\kappa + \frac{V_1}{2} + V_1 + \frac{V_0AB\delta}{\alpha} \\
& + (2n+1) \left(\sqrt{\frac{1}{4} + V_1^2 + V_0B^2\delta + \kappa(\kappa+1) + 2V_1\kappa + V_1} + \sqrt{\frac{M^2}{4\alpha^2} + \frac{V_0A^2\delta}{4\alpha^2} - \frac{(E_{n\kappa}^s)^2}{4\alpha^2} - \frac{MC_s}{4\alpha^2} + \frac{E_{n\kappa}^s C_{ps}}{4\alpha^2}} \right) \\
& + 2 \sqrt{\left(\frac{M^2}{4\alpha^2} + \frac{V_0A^2\delta}{4\alpha^2} - \frac{(E_{n\kappa}^s)^2}{4\alpha^2} - \frac{MC_s}{4\alpha^2} + \frac{E_{n\kappa}^s C_{ps}}{4\alpha^2} \right) \left(\frac{1}{4} + V_1^2 + V_0B^2\delta + \kappa(\kappa+1) + 2V_1\kappa + V_1 \right)} = 0 \quad (52)
\end{aligned}$$

The corresponding components of the upper and lower wave functions are calculated as

$$F_{n\kappa}^s(r) = N_{n\kappa} (e^{-2\alpha r})^{\sqrt{Q_3^s}} (1 - e^{-2\alpha r})^{\frac{1}{2} + \sqrt{\frac{1}{4} + Q_1^s + Q_3^s - Q_2^s}} P_n^{(2\sqrt{Q_3^s}, 2\sqrt{\frac{1}{4} + Q_1^s + Q_3^s - Q_2^s})} (1 - 2e^{-2\alpha r}) \quad (53)$$

$$G_{n\kappa}^s(r) = \frac{1}{M + E_{n\kappa}^s - C_s} \left(\frac{d}{dr} + \frac{\kappa}{r} + V_1 \left(\frac{e^{-\alpha r}}{r} \right) \right) F_{n\kappa}^s(r) \quad (54)$$

Where, $E_{n\kappa}^s \neq C_s - M$.

4. Numerical Results

In order to test the accuracy of our work, we have computed some numerical results given in Tables 1 and 2, where we have used the parameter values as $\alpha = 0.01 \text{ fm}$, $M = 1.0 \text{ fm}^{-1}$, $A = 0.7$, $B = 0.5$, $C_s = 0 \text{ fm}^{-1}$, $V_0 = 1 \text{ fm}^{-1}$ with two values of tensor interaction $V_1 = 0 \text{ fm}^{-1}$ and $V_1 = 0.2 \text{ fm}^{-1}$ for the spin symmetry case and $\alpha = 0.01$, $M = 1.0 \text{ fm}^{-1}$, $A = 0.7$, $B = 0.5$, $C_{ps} = 0 \text{ fm}^{-1}$ with two values of the tensor interaction $V_1 = 0 \text{ fm}^{-1}$ and $V_1 = 0.2 \text{ fm}^{-1}$ for pseudo-spin symmetry case, respectively. In Tables 1 and 2, the numerical result of the energy eigen-value, Eqns. (40) and (52) for the pseudo-spin and spin symmetry limits have been presented in the presence and absence of the tensor interaction potential by using the above given parameters. From Tables 1 and 2, one can see that members of the spin and pseudo-spin doublets have same energy in the absence of tensor interaction ($V_1 = 0$). In addition, it can be seen from Tables 1 and 2 that degeneracies between two states in the spin and pseudo-spin doublets are removed by tensor interaction ($V_1 \neq 0$). The reason can be found by considering the energy eigen-value

Eqns. (40) and (52), which offer different contributions to each energy level in the pseudo-spin and spin doublet [26]. However, there are still extra degeneracies among the energy levels. See Ref. [26] for more details for the presence of these extra degeneracies. Approximate solutions of the Dirac equation is obtained in the absence ($V_1 = 0$) and the presence ($V_1 = 0.2$) of the Yukawa tensor potential for various values of the quantum numbers n and κ . The results are reported in Tables 1 and 2 under the condition of the pseudo-spin and spin symmetries, respectively. Fig. 2 shows the effects of the α -parameter on the bound states for $V_1 = 0.2$. We see that when the α -parameter increases, the bound-states become more bounded in both limits. Fig. 3 presents the dependence of the bound-state energy levels on potential parameter V_0 . It is seen in Fig. 3 that although the bound-states in the pseudo-spin symmetry become more bounded with increasing V_0 , they become less bounded in the other limit. In Fig. 4), we have portrayed the behavior of energy vs. C_{ps} and C_s , where we see that solutions obtained become more bounded with increasing C_s while the other limit solution become less bounded with increasing C_{ps} . In Figs. 5 and 6, the wave functions with and without a tensor interaction are plotted. We see that although the tensor interaction affects the shape of the wave function, it does not change the node structures of the upper and lower components of the Dirac spinors.

5. Conclusions

In this work, we solved the Dirac equation with the generalized Yukawa potential with Yukawa tensor

interaction within the framework of pseudo-spin and spin symmetry limits using a powerful mathematical NU technique. Our results, due to the

choice of the potentials, might be useful in hadronic spectroscopy.

Table 1: Energies in the Pseudo-spin Symmetry Limit for $M = 1\text{fm}^{-1}$, $V_0 = -1\text{fm}^{-1}$, $\alpha = 0.01\text{fm}$, $A = 0.7$, $B = 0.5$, $C_{ps} = 0\text{fm}^{-1}$.

$\tilde{\ell}$	$n, \kappa < 0$	(ℓ, j)	$E_{n\kappa}^{ps}(\text{fm}^{-1})$ ($V_1 = 0$)	$E_{n\kappa}^{ps}(\text{fm}^{-1})$ ($V_1 = 0.2$)	$n - 1, \kappa > 0$	$(\ell + 2, j + 1)$	$E_{n\kappa}^{ps}(\text{fm}^{-1})$ ($V_1 = 0$)	$E_{n\kappa}^{ps}(\text{fm}^{-1})$ ($V_1 = 0.2$)
1	1,-1	$1S_{\frac{1}{2}}$	-0.494486247	-0.490682707	0,2	$0d_{\frac{3}{2}}$	-0.494486247	-0.497246416
2	1,-2	$1P_{\frac{3}{2}}$	-0.504128204	-0.502842352	0,3	$0f_{\frac{5}{2}}$	-0.504128204	-0.505119753
3	1,-3	$1d_{\frac{5}{2}}$	-0.507883658	-0.507339924	0,4	$0g_{\frac{7}{2}}$	-0.507883658	-0.508303507
4	1,-4	$1f_{\frac{7}{2}}$	-0.50946331	-0.509242971	0,5	$0h_{\frac{9}{2}}$	-0.50946331	-0.509624893
1	2,-1	$2S_{\frac{1}{2}}$	-0.503823004	-0.502366404	1,2	$1d_{\frac{3}{2}}$	-0.503823004	-0.50491349
2	2,-2	$2P_{\frac{3}{2}}$	-0.507830896	-0.507267092	1,3	$1f_{\frac{5}{2}}$	-0.507830896	-0.508264425
3	2,-3	$2d_{\frac{5}{2}}$	-0.509451853	-0.509227173	1,4	$1g_{\frac{7}{2}}$	-0.509451853	-0.50961655
4	2,-4	$2f_{\frac{7}{2}}$	-0.509977452	-0.509929545	1,5	$1h_{\frac{9}{2}}$	-0.509977452	-0.50999764
1	3,-1	$3S_{\frac{1}{2}}$	-0.507696899	-0.507060003	2,2	$2d_{\frac{3}{2}}$	-0.507696899	-0.508173554
2	3,-2	$3P_{\frac{3}{2}}$	-0.509430517	-0.509196481	2,3	$2f_{\frac{5}{2}}$	-0.509430517	-0.509601569
3	3,-3	$3d_{\frac{5}{2}}$	-0.509975442	-0.509925493	2,4	$2g_{\frac{7}{2}}$	-0.509975442	-0.509997047

Table 2: Energies in the spin symmetry limit for $M = 1\text{fm}^{-1}$, $V_0 = 1\text{fm}^{-1}$, $\alpha = 0.01\text{fm}$, $A = 0.7$, $B = 0.5$, and $C_s = 0\text{fm}^{-1}$.

ℓ	$n, \kappa < 0$	(ℓ, j)	$E_{n\kappa}^s(\text{fm}^{-1})$ ($V_1 = 0$)	$E_{n\kappa}^s(\text{fm}^{-1})$ ($V_1 = 0.2$)	$n-1, \kappa > 0$	(ℓ, j)	$E_{n\kappa}^s(\text{fm}^{-1})$ ($V_1 = 0$)	$E_{n\kappa}^s(\text{fm}^{-1})$ ($V_1 = 0.2$)
1	0, -2	$0P_{\frac{3}{2}}$	0.466208291	0.452622801	0,1	$0P_{\frac{1}{2}}$	0.466208291	0.475855155
2	0, -3	$0d_{\frac{5}{2}}$	0.495260472	0.4921033	0,2	$0d_{\frac{3}{2}}$	0.495260472	0.497864891
3	0, -4	$0f_{\frac{7}{2}}$	0.504248294	0.50308144	0,3	$0f_{\frac{5}{2}}$	0.504248294	0.5052624
4	0, -5	$0g_{\frac{9}{2}}$	0.507912105	0.50741234	0,4	$0g_{\frac{7}{2}}$	0.507912105	0.508351028
1	1, -2	$1P_{\frac{3}{2}}$	0.494486247	0.490851104	1,1	$1P_{\frac{1}{2}}$	0.494486247	0.49736420
2	1, -3	$1d_{\frac{5}{2}}$	0.504128204	0.502915806	1,2	$1d_{\frac{3}{2}}$	0.504128204	0.505174398
3	1, -4	$1f_{\frac{7}{2}}$	0.507883658	0.507375081	1,3	$1f_{\frac{5}{2}}$	0.507883658	0.508329373
4	1, -5	$1g_{\frac{9}{2}}$	0.50946331	0.509258421	1,4	$1g_{\frac{7}{2}}$	0.50946331	0.509635044
1	2, -2	$2P_{\frac{3}{2}}$	0.503823004	0.502443681	2,1	$2P_{\frac{1}{2}}$	0.503823004	0.504969865
2	2, -3	$2d_{\frac{5}{2}}$	0.507830896	0.50730292	2,2	$2d_{\frac{3}{2}}$	0.507830896	0.508290686
3	2, -4	$2f_{\frac{7}{2}}$	0.509451853	0.50924282	2,3	$2f_{\frac{5}{2}}$	0.509451853	0.509626832
4	2, -5	$2g_{\frac{9}{2}}$	0.509977452	0.509933529	2,4	$2g_{\frac{7}{2}}$	0.509977452	0.509998287

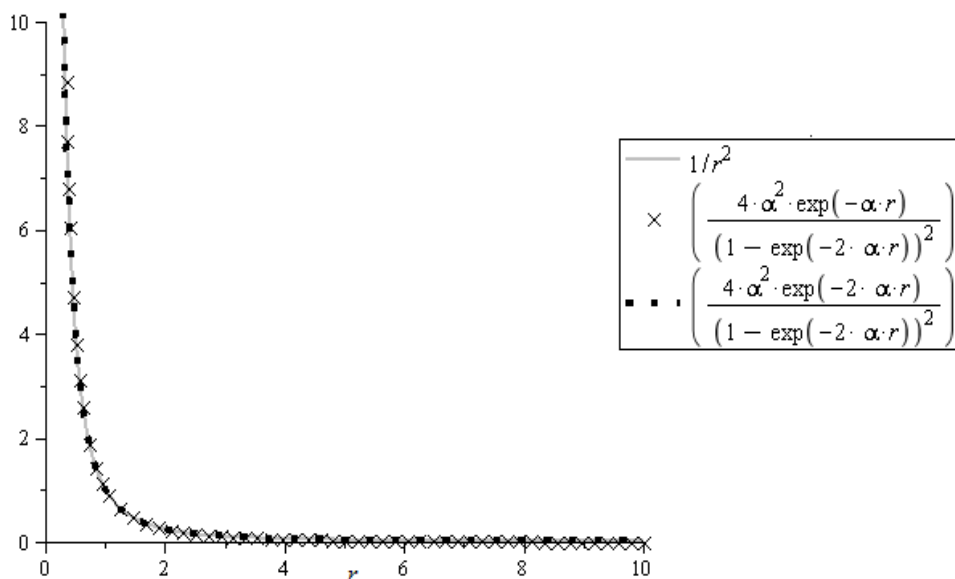


Fig.1: $\frac{1}{r^2}$ and its approximations for $\alpha=0.01$.

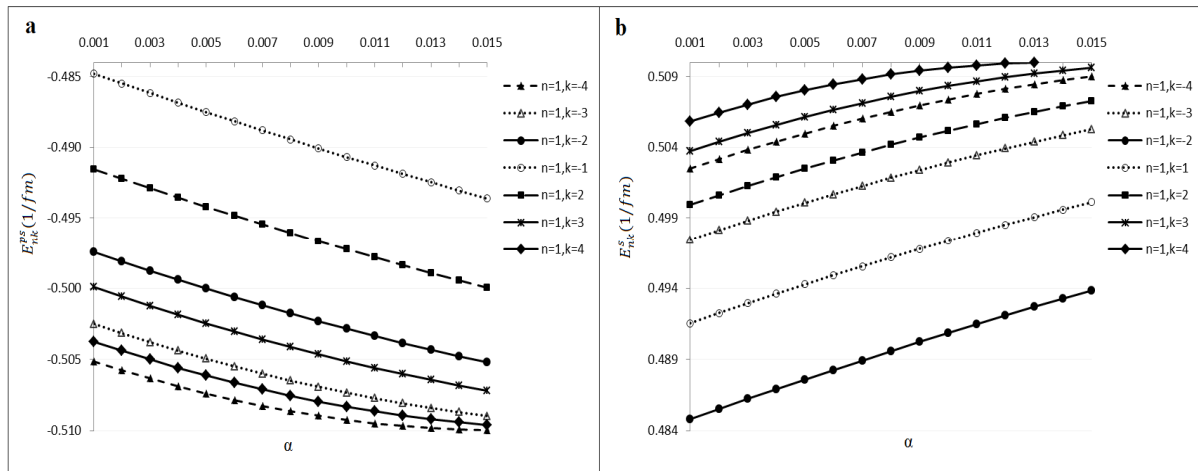


Fig.2: a) Energy vs. α for pseudo-spin symmetry limit for $M = 1fm^{-1}$, $V_1 = 0.2fm^{-1}$, $V_0 = -1fm^{-1}$, $A = 0.7$, $B = 0.5$, and $C_{ps} = 0fm^{-1}$.

b) Energy vs. α for spin symmetry limit for $M = 1fm^{-1}$, $V_1 = 0.2fm^{-1}$, $V_0 = 1fm^{-1}$, $A = 0.7$, $B = 0.5$, and $C_s = 0fm^{-1}$.

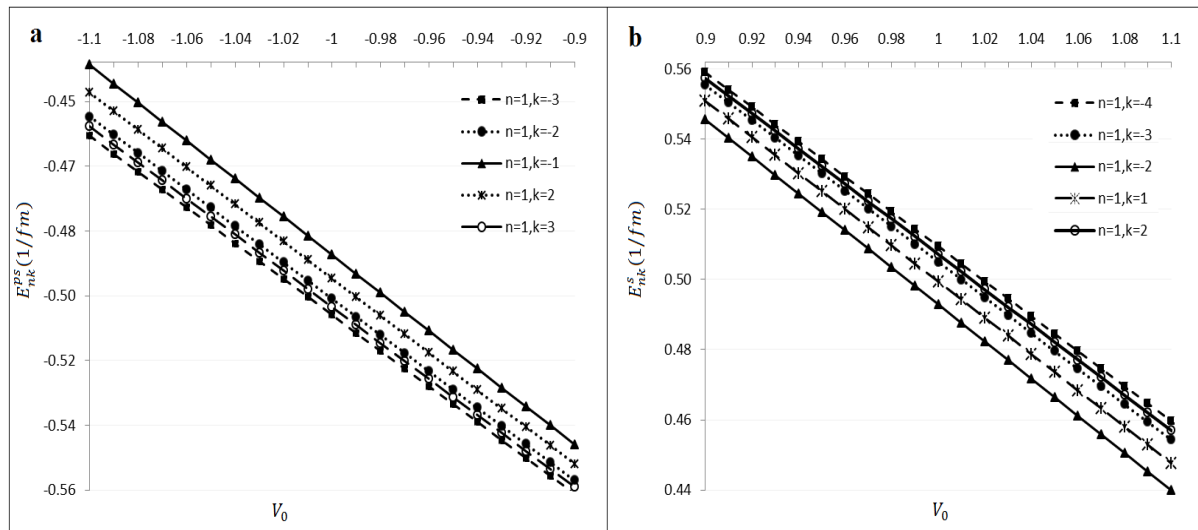


Fig.3: a) Energy vs. V_0 for pseudo-spin symmetry limit for $M = 1fm^{-1}$, $V_1 = 0.2fm^{-1}$, $A = 0.7$, $B = 0.5$, and $C_{ps} = 0fm^{-1}$.

b) Energy vs. V_0 for spin symmetry limit for $M = 1fm^{-1}$, $\alpha = 0.01fm$, $V_1 = 0.2fm^{-1}$, $A = 0.7$, $B = 0.5$, and $C_s = 0fm^{-1}$.

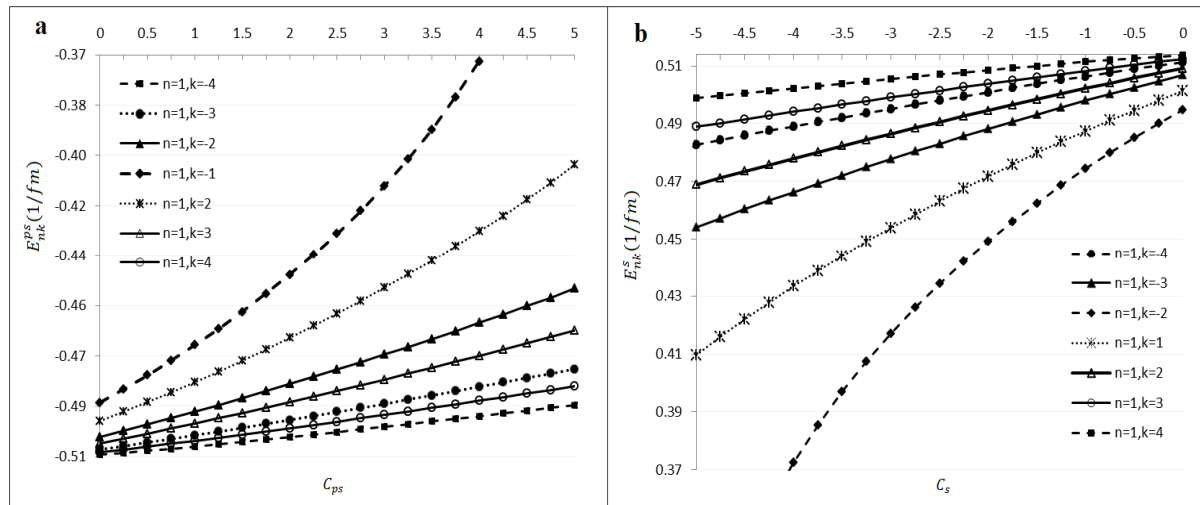


Fig.4: a) Energy vs. C_{ps} for pseudo-spin symmetry limit for $M = 1fm^{-1}$, $V_0 = -1fm^{-1}$, $V_1 = 0.2fm^{-1}$, $\alpha = 0.01fm$, $A = 0.7$, and $B = 0.5$.

b) Energy vs. C_s for spin symmetry limit for $M = 1fm^{-1}$, $V_0 = 1fm^{-1}$, $V_1 = 0.2fm^{-1}$, $\alpha = 0.01fm$, $A = 0.7$, and $B = 0.5$.

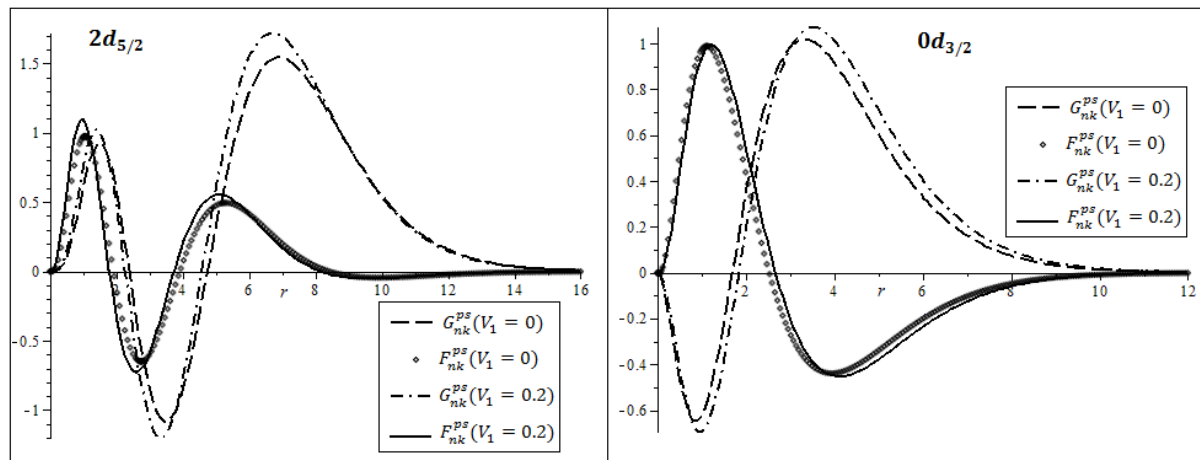


Fig.5: Wavefunction for pseudo-spin symmetry limit for $M = 1fm^{-1}$, $V_0 = -1fm^{-1}$, $\alpha = 0.01fm$, $A = 0.7$, $B = 0.5$, and $C_{ps} = 0fm^{-1}$.

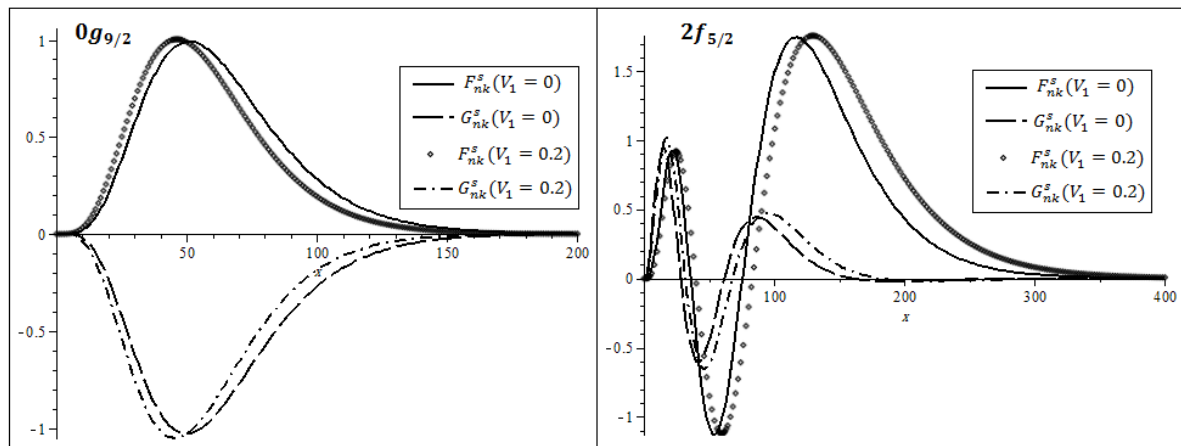


Fig.6: Wavefunction for spin symmetry limit for $M = 1\text{fm}^{-1}$, $V_0 = 1\text{fm}^{-1}$, $\alpha = 0.01\text{fm}$, $A = 0.7$, $B = 0.5$, and $C_s = 0\text{fm}^{-1}$.

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