Higher Dimensional FRW String Cosmological Models in a New Scalar-tensor Theory of Gravitation

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Explicit field equations of a new scalar tensor theory of gravitation proposed by the Sen-Dunn theory are obtained with the help of a five dimensional FRW metric in the context of cosmic strings. Assuming a functional relationship between the metric potentials, the solutions of the field equations are obtained in two cases: (i) geometric strings (i.e., $\lambda = \rho$) and (ii) massive strings (i.e., $\lambda + \rho = 0$). Some physical and geometric properties of the solutions are also discussed in each case.

1. Introduction

It is well known that a gravitational scalar field, beside the metric of the space-time, must exist in the frame work of the present unified theories. Hence, there has been much interest in scalar tensor theories of gravitation. Several theories are proposed as alternatives to Einstein's theory to reveal the nature of the universe in the early stage of evolution. The most important among them were scalar-tensor theories proposed by Lyra [1], Brans-Dicke [2], Nordtvedt [3], Wagoner [4] and Saez and Ballester [5]. Sen and Dunn [6] have proposed a new scalar-tensor theory of gravitation in which both the scalar and tensor fields have intrinsic geometrical significance. The scalar field in this theory is characterized by the function $\phi = \phi(x^i)$ where x^i are coordinates in the four–dimensional Lyra manifold and the tensor field is identified with the metric tensor g_{ij} of the manifold. The field equations given by Sen and Dunn [6] for the combined scalar and tensor fields are

$$R_{ij} - \frac{1}{2} g_{ij} R = \omega\phi^{-2}(\phi_{,i}\phi_{,j} - \frac{1}{2} g_{ij}\phi_{,k}\phi^{,k}) - \phi^{-2}T_{ij} \quad (1)$$

Where, ω is a dimensionless constant, R_{ij} and R are the usual Ricci-tensor and Riemann-curvature scalar (in our units, $C = 8\pi G = 1$), respectively.

In view of Kaluza-Klein theories [7–10], the study of higher-dimensional cosmological models acquired great significance. An interesting possibility, known as the cosmological dimensional reduction process, is based on the idea that at the very early stage all dimensions in the universe are comparable. Later, the scale of the extra dimensions becomes so small as to be unobservable by experiencing contraction. This process was first proposed by Chodos and Detweiler [11] who showed that, in the framework of pure gravitational theory of Kaluza-Klein, the extra dimension contracts to a very small scale, while the other spatial dimensions expand isotropically. Guth [12] and Alvarez and Gavela [13] observed that during the contraction process, extra dimensions produce large amount of entropy. The geometrization of gravitation by Einstein in his general theory of relativity inspired several authors to geometrize other physical fields. Weyl [14] proposed a unified theory to geometrize gravitation and electro-magnetism.

The study of string theory is important in the early stages of the evolution of the Universe before the creation of particles. Cosmic strings have received considerable attention in cosmology as they are believed to give rise to density perturbations leading to the formation of galaxies [15]. The general relativistic treatment of strings was initiated by Letelier [16,17] and Stachel [18]. Reddy [19], Reddy and Rao [20], Rahaman [21], Rahaman et al. [22], Venkateswarlu, and Pavan Kumar [23], Mohanty et al. [24,25] Singh et al. [26], Pradhan [27-29], Pradhan and Mathur,[30] Pradhan et al. [31,32] and Katore et al. [33] are some of the authors who have studied various aspects of string cosmologies in theory of relativity as well as in alternate theories of gravitation.

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The strings that form the cloud are massive strings instead of geometric strings. Each massive string is formed by a geometric string with particles attached along its extension. This is simplest model wherein we have particles and strings together. In principle we can eliminate the strings and end up with a cloud of particles. This is a desirable property of a model of a string cloud to be used in cosmology since strings are not observed at the present time of evolution of the universe. Motivated by the situations discussed above, in this paper, we investigate the five dimensional FRW model in a new scalar-tensor theory of gravitation proposed by Sen and Dunn [6] in the presence of cosmic strings. Sec. 2 contains the five dimensional FRW metric and the field equations of this theory. In Sec. 3, we present the solution of field equations obtained in the context of cosmic strings and also discussed some properties of the models. The last section contains concluding remarks on the models.

2. Metric and Field Equations

Here we consider the five dimensional FRW metric of the form

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right]^2$$

$$+ A^2(t) d\mu \tag{2}$$

Where, $R(t)$ is the scale factor and k = 0, -1 or +1 is the curvature parameter for flat, open and closed universe, respectively. The fifth coordinate μ is also assumed to be space like coordinate.

The energy momentum tensor for a cloud of massive strings that can be written as

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j \tag{3}$$

Here ρ is the rest energy density of the cloud of strings with particles attached to them, λ is the tension density of the strings and $\rho = \rho_p + \lambda$, ρ_p being the energy density of the particles. The velocity u^i describes the five – velocity, which has components (1, 0, 0, 0,0) for a cloud of particles and x^i represents the direction of string which will satisfy

$$u^i u_i = -x^i x_i = 1 \text{ and } u^i x_i = 0 \tag{4}$$

The direction of the strings is taken to be along X_4 – axis so that we have X^i = (0,0,0,0, 1/A). Now, the field equations for the metric (Eqn. (4)) can be written as

$$3\frac{\ddot{R}}{R} + 3\frac{\dot{R}}{R^2}^2 + 3\frac{k}{R^2} = \lambda\phi^{-2} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi^2}^2\right) \tag{5}$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}}{R^2}^2 + 2\frac{\dot{R}\dot{A}}{RA} + \frac{\ddot{A}}{A} + \frac{k}{R^2} = \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi^2}^2\right) \tag{6}$$

$$3\frac{\dot{R}}{R^2}^2 + 3\frac{\dot{R}\dot{A}}{RA} + 3\frac{k}{R^2} = \rho\phi^{-2} - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi^2}^2\right) \tag{7}$$

Where, the overhead ‘dot’ denotes ordinary differentiation with respect to ‘t’.

3. Solutions to the Field Equations

The field equations (5)-(7) are a system of three equations with five unknown parameters A, R, ϕ, ρ and λ. We need two additional conditions to get a deterministic solution of the above system of equations. Thus we present the solutions of the field equations in the following physically meaningful cases:

(i) The simplest relation between ρ and λ is the proportionality relation, written as $\rho = \beta\lambda$ where β is a proportionality constant which gives rise to the following three cases:

a. For $\beta = 1$, we get geometric strings (or) Nambu strings
b. For $\beta = -1$, we get massive strings

and (ii) a functional relationship [34] between the metric functions A and R of the form

$$A = R^m \tag{8}$$

Where, ‘m’ is an arbitrary constant.

3.1. Geometric strings (or) Nambu strings (the case $\rho = \lambda$)

Now the field equations (5)-(7) together with (8) reduces to

$$\frac{\ddot{R}}{R} + (m+2)\frac{\dot{R}}{R^2}^2 + \frac{2k}{(2m+1)R^2} = 0 \tag{9}$$

Case 3.1.1: Flat model ($k=0$)

In this case the field equation (9) admits the solution

$$R(t)=[(m+3)(at+b)]^{\frac{1}{(m+3)}}\ldots \tag{10}$$

Where, 'a' and 'b' are integrating constants. Thus the general solution of the field equations (5)-(7) can be written as

$$R(t)=[(m+3)(at+b)]^{\frac{1}{(m+3)}}$$

$$A(t)=[(m+3)(at+b)]^{\frac{m}{(m+3)}} \tag{11}$$

and the scalar field is given by

$$\phi=\phi_0(c_1t+c_2)^{p_1} \tag{12}$$

Where, ϕ_0 is an arbitrary constant and $p_1=\left[\frac{-2(3+3m)}{\omega(m+3)^2}\right]^{\frac{1}{2}}$, $m<-1$. By making use of Eqn. (11) in Eqns. (5)-(7), the string energy density ρ and the tension density λ become zero. Hence, it is observed that cosmic geometric strings do not co-exist with the scalar field in this theory.

Case 3.1.2: Closed model ($k=1$)

From Eqn. (9), we have

$$\frac{\ddot{R}}{R}+(m+2)\frac{\dot{R}^2}{R^2}+\frac{2}{(2m+1)R^2}=0 \tag{13}$$

whose first integral is

$$\dot{R}^2=-\frac{4}{(2m+4)(2m+1)}+d\,R^{-(2m+4)} \tag{14}$$

Where, '*d*' is an integrating constant. Eqn. (14) admits a closed form solution only if *d* = 0, and hence its solution is given by

$$R(t)=\frac{2t+c_3\sqrt{n}}{\sqrt{n}} \tag{15}$$

Consequently, we have

$$A(t)=\left(\frac{2t+c_3\sqrt{n}}{\sqrt{n}}\right)^m \tag{16}$$

together with the scalar field

$$\phi(t)=\phi_0(2t+c_3)^{p_3} \tag{17}$$

Where, c_3 is an integrating constant and $p_3=\sqrt{\frac{-3m}{\omega}}$; $n=-(m+2)(4m+2)$, $-2<m<-\frac{1}{2}$.

Thus the metric (Eqn. (4)) in this case takes the form

$$ds^2=-dt^2+\left(\frac{2t+c_3\sqrt{n}}{\sqrt{n}}\right)^2\left[\frac{dr^2}{1-r^2}+r^2d\theta^2+r^2\sin^2\theta\,d\phi^2\right]+\left(\frac{2t+c_3\sqrt{n}}{\sqrt{n}}\right)^{2m}d\mu^2 \tag{18}$$

The string energy density ρ and tension density λ are

$$\rho=\lambda=-12\phi_0^{\,2}m(m+2)(2t+c_3\sqrt{n})^{(2p_3-2)} \tag{19}$$

The energy conditions viz., $\rho>0$, and $\lambda>0$ are identically satisfied only for $-2<m<-\frac{1}{2}$. When *m* = -1, Eqns. (21)-(23) reduces to

$$R(t)=(c_3+\sqrt{2}t)$$

$$A(t)=(c_3+\sqrt{2}t)^{-1} \tag{20}$$

The corresponding scalar field takes the form

$$\phi(t)=\phi_0(c_1t+c_2)^{\sqrt{2}} \tag{21}$$

The kinematical parameters are

$$\rho=\lambda=12\phi_0^{\,2}(2t+c_3\sqrt{2})^{2(\sqrt{2}-1)} \tag{22}$$

$$\theta = \frac{2\sqrt{2}}{(c_3 + \sqrt{2}t)}$$

$$\sigma = \sqrt{\frac{56}{9}} \frac{1`}{(\sqrt{2}t + c_3)} \qquad (23)$$

$$V^4 = \sqrt{-g} = \left(\sqrt{2}t + c_3\right)^2$$

$$q = \frac{1}{2}$$

In this case also we observe that the extra dimension $A(t)$ contracts to a small size while $R(t)$ expands with an increase in time. Since $q > 0$, for $m = -1$, the 'closed' model exhibits the non-existence of inflation in Sen-Dunn's theory of gravitation. The 'closed' FRW model also possesses a line singularity as $\theta \to 0, V \to \infty, R(t) \to \infty, A(t) \to 0$ when $t \to \infty$. The volume is increasing indefinitely with the increase in time.

Case 3.1.3: Open model (k = -1)

Here the field equations (5)-(7) together with (8) reduce to

$$\frac{\ddot{R}}{R} + (m+2)\frac{\dot{R}}{R^2}^2 - \frac{2}{(2m+1)R^2} = 0 \qquad (24)$$

The first integral of above equation is

$$\dot{R}^2 = \frac{4}{(2m+4)(2m+1)} + e\,R^{-(2m+4)} \qquad (25)$$

Since we are looking for a closed form solution of Eqn. (25), we take e = 0 and hence Eqn. (25) on integration yields

$$R(t) = \frac{2t + c_4\sqrt{(m+2)(4m+2)}}{\sqrt{(m+2)(4m+2)}}$$

and (26)

$$A(t) = \left(\frac{2t + c_4\sqrt{(m+2)(4m+2)}}{\sqrt{(m+2)(4m+2)}}\right)^m$$

The scalar field is obtained as

$$\phi = \phi_0\left(2t + c_4\sqrt{(m+2)(4m+2)}\right)^{p_4} \qquad (27)$$

Where, c_4 is an integrating constant and $p_4 = \sqrt{\frac{-3m}{\omega}}$, $m < 0$.

Thus the metric (Eqn. (4)), for open FRW model, takes the form

$$ds^2 = -dt^2 + \left[\frac{2t + c_4\sqrt{(m+2)(4m+2)}}{\sqrt{(m+2)(4m+2)}}\right]^2 \left[\frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right] + \left(\frac{2t + c_4\sqrt{(m+2)(4m+2)}}{\sqrt{(m+2)(4m+2)}}\right)^{2m} d\mu^2 \qquad (28)$$

The string energy density ρ and tension density λ are

$$\rho = \lambda = -\phi_0{}^2 12m(m+2)(2t + c_4\sqrt{(m+2)(4m+2)})^{(2p_4 - 2)} \qquad (29)$$

The energy conditions viz., $\rho > 0$ and $\lambda > 0$ are identically satisfied for $-2 < m < 0$. If $m = -\frac{1}{4}$, we have

$$R(t) = \frac{(\sqrt{7}c_4 + 4t)}{\sqrt{7}}$$

$$A(t) = \left[\frac{(\sqrt{7}c_4 + 4t)}{\sqrt{7}}\right]^{\frac{-1}{4}} \qquad (30)$$

$$\phi(t) = \phi_0(2t + \frac{\sqrt{7}}{2}c_4)^{\frac{1}{\sqrt{2}}} \qquad (31)$$

$$\rho = \lambda = \phi_0{}^2 \frac{21}{4}(2t + \frac{\sqrt{7}}{2}c_4)^{(\sqrt{2}-2)} \qquad (32)$$

And the kinematical parameters are given by

Scalar expansion $\theta = \frac{11}{\left(4t + c_4\sqrt{7}\right)}$

Shear scalar $\sigma = \frac{13}{3\sqrt{2}} \frac{1}{\left(4t + c_4\sqrt{7}\right)}$ (33)

Spatial volume $V^4 = \sqrt{-g} = \left(\frac{\left(4t + c_4\sqrt{7}\right)}{\sqrt{7}}\right)^{\frac{11}{4}}$

Deceleration parameter $q = \frac{-a\,\ddot{a}}{\dot{a}^{\,2}} = \frac{1}{11}$

Thus extra dimension $A(t)$ contracts whereas $R(t)$ expands indefinitely with time if $m < -2$. The 'open' model also possesses a line singularity as $\theta \to 0, V \to \infty, R(t) \to \infty, A(t) \to 0$ when $t \to \infty$. The volume is increasing indefinitely with the increase in time. Since $q > 0$, there is no inflation in this model.

3.2. Massive strings ($\rho + \lambda = 0$) i.e., when $\beta = -1$

In this case we consider Eqns. (5)-(7) together with Eqn. (8) yields

$$\frac{\ddot{R}}{R} + (m+2)\frac{\dot{R}^{\,2}}{R^2} + \frac{2k}{R^2} = 0 \tag{34}$$

Case 3.2.1: Flat model

When $k = 0$, Eqn. (34) on integration yields

$$R(t) = [(m+3)(c_5 t + c_6)]^{\frac{1}{(m+3)}} \tag{35}$$

and

$$A(t) = [(m+3)(c_5 t + c_6)]^{\frac{m}{(m+3)}} \tag{36}$$

The scalar field is obtained as

$$\phi = \phi_0 (c_5 t + c_6)^{\,p_5} \tag{37}$$

Where, c_5 ,and c_6 are arbitrary constants, and

$p_5 = \left[\frac{-6(m+1)}{\omega(m+3)^2}\right]^{\frac{1}{2}}$.

By making use of Eqns. (35) and (36) in Eqns. (5)-(7), the string energy density ρ and the tension density λ become zero. Hence it is observed that cosmic massive strings do not co-exist with the scalar field in this theory.

***Case* 3.2.2: Closed model** $k = 1$

In this case Eqn. (34) reduces to the form

$$\frac{\ddot{R}}{R} + (m+2)\frac{\dot{R}^{\,2}}{R^2} + \frac{2}{R^2} = 0 \tag{38}$$

which admits the solution

$$R(t) = \left(\frac{\sqrt{2}t + c_7\sqrt{-(m+2)}}{\sqrt{-(m+2)}}\right)$$

$$A(t) = \left(\frac{\sqrt{2}t + c_7\sqrt{-(m+2)}}{\sqrt{-(m+2)}}\right)^m, m < -2 \tag{39}$$

$$\phi = \phi_0 (\sqrt{2}t + c_7\sqrt{n})^{\,p_6} \tag{40}$$

Where, c_7 is arbitrary constant and

$p_6 = \left[\frac{m(2m+1)}{\omega}\right]^{\frac{1}{2}}$.

Thus the metric (Eqn. (4)) takes the form

$$ds^2 = -dt^2 + \left(\frac{\sqrt{2}t + c_7\sqrt{-(m+2)}}{\sqrt{-(m+2)}}\right)^2 \left[\frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right] + \left(\frac{\sqrt{2}t + c_7\sqrt{-(m+2)}}{\sqrt{-(m+2)}}\right)^{2m} d\mu^2 \tag{41}$$

and the corresponding scalar field is given by Eqn. (46).

The string energy density ρ, tension density λ are given by

$$\rho = -\lambda = \phi_0^{\,2}\, 2m(m+2)(\sqrt{2}t + c_7\sqrt{n})^{(2p_6 - 2)} \tag{42}$$

and the particle density ρ_p, the scalar expansion θ, the shear scalar σ, spatial volume V and the deceleration parameter q are given by

$$\rho_p = 4\phi_0{}^2 m(m+2)(\sqrt{2}t + c_7\sqrt{n})^{(2p_6-2)}$$

$$\theta = \frac{\sqrt{2}(m+3)}{(\sqrt{2}t + c_7\sqrt{n})}$$

$$\sigma = \left[\frac{1}{9}\frac{(7m^2-12m+9)}{(\sqrt{2}t + c_7\sqrt{n})^2}\right]^{\frac{1}{2}} \quad (43)$$

$$V^4 = \sqrt{-g} = \left(\frac{\sqrt{2}t + c_7\sqrt{n}}{\sqrt{n}}\right)^{m+3}$$

$$q = \frac{-a\ddot{a}}{\dot{a}^2} = -\frac{m}{(m+3)}$$

Again the extra dimension $A(t)$ is amenable for contraction if $m < -2$. The dominant energy conditions implies that $\rho > 0$ and $\rho^2 \geq \lambda^2$. These energy conditions do not restrict the sign of λ, accordingly the expressions given by Eqn. (42) satisfies all these conditions. We observe that $\frac{\rho_p}{|\lambda|} = 2$. Since $\frac{\rho_p}{|\lambda|} > 1$, thus we may conclude that the particles dominate over the strings in this model. This model has no singularity at t = 0 and undergoes expansion with time.

Case 3.2.3: Open model $k = -1$

In this case Eqn. (34) reduces to the form

$$\frac{\ddot{R}}{R} + (m+2)\frac{\dot{R}}{R^2}^{2} - \frac{2}{R^2} = 0 \quad (44)$$

Eqn. (50) on integration yields

$$R(t) = \left(\frac{\sqrt{2}t + c_8\sqrt{(m+2)}}{\sqrt{(m+2)}}\right) \quad (45)$$

$$A(t) = \left(\frac{\sqrt{2}t + c_8\sqrt{(m+2)}}{\sqrt{(m+2)}}\right)^m \quad (46)$$

$$\phi = \phi_0(\sqrt{2}t + c_8\sqrt{(m+2)})^{p_7} \quad (47)$$

Where, c_8 is arbitrary constant and

$$p_7 = \left[\frac{m(2m+1)}{\omega}\right]^{\frac{1}{2}}.$$

Thus the metric (4) takes the form

$$ds^2 = -dt^2 + \left(\frac{\sqrt{2}t + c_8\sqrt{(m+2)}}{\sqrt{(m+2)}}\right)^2\left[\frac{dr^2}{1+r^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right] + \left(\frac{\sqrt{2}t + c_8\sqrt{(m+2)}}{\sqrt{(m+2)}}\right)^{2m} d\mu^2 \quad (48)$$

The string energy density ρ, tension density λ are given by

$$\rho = -\lambda = \phi_0{}^2 2m(m+2)(\sqrt{2}t + c_8\sqrt{(m+2)})^{(2p_7-2)} \quad (49)$$

The particle density ρ_p, the scalar expansion θ, the shear scalar σ, spatial volume V and the deceleration parameter q are given by

$$\rho_p = 4\phi_0{}^2 m(m+2)(\sqrt{2}t + c_8\sqrt{n})^{(2p_7-2)}$$

$$\theta = \frac{\sqrt{2}(m+3)}{(\sqrt{2}t + c_7\sqrt{n})}$$

$$\sigma = \left[\frac{1}{9}\frac{(7m^2-12m+9)}{(\sqrt{2}t + c_7\sqrt{n})^2}\right]^{\frac{1}{2}} \quad (50)$$

$$V^4 = \sqrt{-g} = \left(\frac{\sqrt{2}t + c_8\sqrt{n}}{\sqrt{(m+2)}}\right)^{m+3}$$

$$q = \frac{-a\ddot{a}}{\dot{a}^2} = -\frac{m}{(m+3)}$$

From Eqns. (50) and (51), it is clear that the scale factors $R(t)$ and $A(t)$ increase indefinitely. Thus there is no compactification of extra dimension in open FRW string cosmological model. Further the energy density ρ, tension density λ of the universe is positive throughout the evolution except

$-2 < m < 0$. This model is free from initial singularity but possesses line singularity at $t = -\frac{c_8}{\sqrt{2}}\sqrt{(m+2)}$ The energy density of the universe increases with increase of cosmic time *t*.

4. Conclusions

In this paper, we have considered the higher dimensional Friedmann-Robertson-Walker space-time in Sen-Dunn theory of gravitation in the context of cosmic strings. The solution of the field equations are obtained and discussed in three different scenarios, namely, (i) geometric strings (i.e., $\lambda = \rho$) and (ii) massive strings (i.e., $\lambda + \rho = 0$). It is observed that the power index of *m* of the metric potential has a range of values for which the scalar field is real. Since *m* has a specific range of values, it is noticed that the extra dimension is amenable for reduction. Thus we may conclude that the scalar field in Sen-Dunn theory plays a significant role for the reduction of extra dimension in the context of cosmic strings.

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