The Remainder of Green Function Perturbation Series Expressed in Closed Form

Hassen Jallouli*
Département de Physique, Faculté des Sciences de Gafsa, Campus Zarroug, Gafsa, Tunisia

Using the technique of integration by parts in functional integrals, we are able to derive the perturbation series of the Green function of φ^4 theory. The advantage of this method in deriving the perturbation series is the fact that we obtained a closed expression for the remainder. This can be very useful in estimating the contribution of higher order diagrams in Green functions.

1. Introduction

Feynman's path integrals [1,2] or functional integrals are an important tool in physics. This formalism has been successfully applied in the quantization procedure of all type of systems, from the simple quantum mechanical problem of a single particle moving in one dimension to more complicated objects such as fields and strings [3]. The use of functional integral is one of the most powerful techniques in quantizing field theories. In this formalism, symmetries of physical systems are easily treated and applied to various problems [4]. The parallel application of this formalism in both high energy and condensed matter physics makes it an important general tool [5]. The analytical and numerical approaches to path integrals are by now central to the development of many other areas of physics, chemistry and materials science, as well as to mathematics and finance [6]. Nevertheless, it has the drawback of being very difficult to carry out calculations and its mathematical foundation is not yet completely clarified [7,8]. The weak perturbation theory generally proves to be insufficient to extract all the physics. A well-known case is given by quantum chromodynamics, which due to its strength of the coupling constant at low energies makes known perturbation techniques useless and demanding the need for non-perturbative solutions.

One of the ideas to surmount the weakness of perturbative method of calculation was proposed in 1977, when Lipatov published his method [9] as a tool for calculating high-order terms in perturbation series and making quantitative estimates for its divergence. The key to Lipatov's method is to express the n^{th} term of the perturbation series as a functional integral and use a saddle point approximation (with respect to the field and the coupling constant) to give an estimate of it.

We present here how we can use the technique of integration by parts in functional integral for the derivation of the perturbation series of the Green function of φ^4 theory. This allowed us to obtain a closed expression for the remainder, which can be subject to non perturbative estimation.

2. Derivation of the Perturbation Series

In the following, we will show how we can derive the perturbation series of the propagator in a simple way using the method of integration by part. We will illustrate this technique on φ^4 theory in Euclidean space.

The connected propagator is given by:

$$G(x,x') = \frac{1}{N}\int D\varphi\, \varphi(x)\varphi(x')e^{-S} \qquad (1)$$

Where the normalization factor N is

$$N = \int D\varphi e^{-S}$$

and S is the action

$$S = \frac{1}{2}\int(\varphi.\Delta.\varphi + \frac{\lambda}{4!}\varphi^4) \qquad (2)$$

Δ is the quadratic term in the action.

$$\frac{\partial}{\partial\varphi(x)} exp\left(-\frac{1}{2}\int \varphi.\Delta.\varphi\right) = -\Delta_{xy}.\varphi(y)\ exp\left(-\frac{1}{2}\int \varphi.\Delta.\varphi\right) \qquad (3)$$

*hassen.jallouli@fsgf.rnu.tn

In Eqn. (3), the integration with respect to y of the second member is to be understood. We can write

$$\varphi(x)\ exp\left(-\frac{1}{2}\int \varphi.\Delta.\varphi\right) = -\Delta_{xy}^{-1}\cdot\frac{\partial}{\partial\varphi(y)}exp\left(-\frac{1}{2}\int\varphi.\Delta.\varphi\right) \tag{4}$$

With this trick, we can write

$$G(x,x') = -\frac{1}{N}\Delta_{xy}^{-1} \cdot\int D\varphi\left(\frac{\partial}{\partial\varphi(y)}exp\left(-\frac{1}{2}\int\varphi.\Delta.\varphi\right)\right)\varphi(x')\ exp\left(-\frac{\lambda}{2.4!}\int\varphi^4\right) \tag{5}$$

We can now integrate by parts with respect to φ (*x*). The boundary term resulting from this integration by parts vanishes because of the exponential of the action, which vanishes for large field. We get

$$G(x,x') = \frac{1}{N}\Delta_{xy}^{-1} \cdot\int D\varphi\ exp\left(-\frac{1}{2}\int\varphi.\Delta.\varphi\right)\frac{\partial}{\partial\varphi(y)}\left(\varphi(x')\ exp\left(-\frac{\lambda}{2.4!}\int\varphi^4\right)\right) \tag{6}$$

We can now do the derivation in this integral and it yields two terms

$$G(x,x') = \Delta^{-1}(x-x') - \frac{\lambda}{12}\frac{1}{N}\Delta_{xy}^{-1}\cdot\int D\varphi\ \varphi^3(y)\varphi(x')\ e^{-S} \tag{7}$$

The first term is nothing but the free propagator. It is the first term of perturbation series. If we make the same manipulation on the second term we will get the other terms of perturbation series.

We write for this

$$\Delta_{xy}^{-1}\cdot\int D\varphi\ \varphi^3(y)\varphi(x')\ e^{-S} = -\Delta_{xy}^{-1}\Delta_{x'y'}^{-1} \cdot\int D\varphi\left(\frac{\partial}{\partial\varphi(y')}exp\left(-\frac{1}{2}\int\varphi.\Delta.\varphi\right)\right)\varphi^3(y)\ exp\left(-\frac{\lambda}{2.4!}\int\varphi^4\right) \tag{8}$$

We can now use integration by parts. But before that we need to use first integration by parts to get the second term of the perturbation series because we have now three fields in the functional integral. We obtain finally

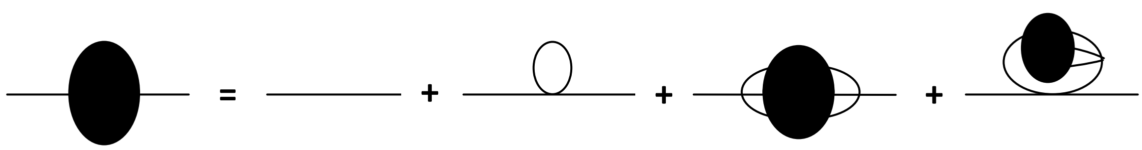
$$G(x,x') = \Delta^{-1}(x-x') - \frac{\lambda}{4}\Delta_{xy}^{-1}.\Delta_{x'y}^{-1}.\Delta_{yy}^{-1} + \frac{1}{N}\left[\frac{\lambda^2}{12^2}\Delta_{xy}^{-1}\Delta_{x'y'}^{-1}.\int D\varphi\ \varphi^3(y)\ \varphi^3(y')e^{-S} + \frac{\lambda^2}{12^2}\Delta_{xy}^{-1}.\Delta_{x'y}^{-1}.\Delta_{yy'}^{-1}\int D\varphi\ \varphi(y)\ \varphi^3(y')e^{-S}\right] \tag{9}$$

3. The Remainder of the Perturbation Series

The first term in Eqn. (8) is the free propagator and the second term is the tadpole term of perturbation series. The term between brackets is the remainder of the series of the second order. It can be expressed in terms of two Green functions of sixth and fourth orders, as can be seen from Eqn. (9).

$$G(x,x') = \Delta^{-1}(x-x') - \frac{\lambda}{4}\Delta_{xy}^{-1}.\Delta_{x'y}^{-1}.\Delta_{yy}^{-1} + \left[\frac{\lambda^2}{12^2}\Delta_{xy}^{-1}\Delta_{x'y'}^{-1}.G^6(y,y,y,y',y',y') + \frac{\lambda^2}{12^2}\Delta_{xy}^{-1}.\Delta_{x'y}^{-1}.\Delta_{yy'}^{-1}.G^4(y,y',y',y')\right] \tag{10}$$

This can be represented schematically by the following diagram



This diagram tells us that Feynman diagrams that represent the remainder can be divided in two classes. Any diagram of the first class has always a sub-diagram of the Green function of the sixth order and any diagram of the second class has always a sub-diagram of the Green function of the fourth order.

Since the perturbation terms are divergent we need to use the renormalization method. We have to introduce some regularization scheme in calculating the perturbation terms, and we have to add counter-terms to the action. The remainder has to be calculated with the action containing these counter-terms.

Eqn. (10) can be transformed to a Dyson-Schwinger type equation relating the two-point Green function to Green functions of sixth and fourth order. This can be done at any order. This can be useful in non-perturbative calculation. We can also use directly the functional expression for the remainder in Eqn. (9) for non-perturbative estimate using, for example, a saddle point approximation as in the Lipatov method [9]. The advantage of formulae in Eqn. (10), even though the remainder is still written as a functional integral, is the fact that we isolate the perturbative part of the Green function. This can be very useful for theories like QCD, where the Green function at short distances is dominated by perturbative terms, and for long distance it is dominated by non-perturbative terms (the remainder). Calculations in this direction are in progress.

4. Conclusion

We derived the perturbation series of Green functions in a simple way using integration by parts. This allowed us to express the remainder in a closed form, which can be the starting point for non-perturbative estimation. A saddle point approximation is suitable to estimate the remainder as in the Lipatov method.

Acknowledgments

The author would like to thank H. Sazdjian, who read a preliminary version of the manuscript and made important remarks and suggestions. Thanks also to the Groupe de Physique Theorique de L'IPN d'Orsay for scientific stay.

References

[1] R. P. Feynman, Rev. Modern Phys. **20**, 376 (1948).

[2] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw–Hill, New York, 1965).

[3] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw–Hill, New York, 1980).

[4] S. Coleman, *Aspects of Symmetry* (Cambridge Univ. Press, Cambridge, 1985).

[5] G. Parisi, *Statistical Field Theory* (Addison–Wesley, New York, 1988).

[6] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets* (World Scientific, Singapore, 2004).

[7] C. De Witt and J. De Witt, Math. Phys. **36**, (1995).

[8] R. H. Cameron, J. Math. and Phys. **39**, 126 (1960).

[9] L. N. Lipatov, Zh. Éksp. Teor. Fiz. **72**, 411 (1977) [Sov. Phys. JETP **45**, 216 (1977)].

Received: 11 July, 2012
Accepted: 4 September, 2012