

KdV Hierarchies on the $2d$ -torus

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In the present contribution, following the previous work on extended higher spin symmetries on the torus, we focus on setting up the integrable KdV hierarchies on $T^2 = S^1 \times S^1$. In fact, two particular systems are considered, namely the KdV and the Burgers nonlinear integrable model associated with the currents of conformal weights $(2, 2)$ and $(1, 1)$, respectively. One key step towards proving the integrability of these systems is to find their Lax pair operators. This is explicitly done and a mapping between the two systems is discussed.

1. Introduction

Integrable systems [1–6] deal with nonlinear differential Equations, which can be solved explicitly or by using the inverse scattering method based on the Lax formulation [1, 2]. On one hand, the particularity of $2d$ integrable systems is due to their pioneering role that they deserve to the KdV differential equation. On the other hand, it is due to the strong connection that exists with conformal symmetries [7] and their higher spin extensions [8, 9]. Since more spectacular realizations are done for the integrability of KdV hierarchies in the $\text{diff}(S^1)$ case, we focus in this work on the study of some properties related to KdV hierarchies in the $\text{diff}(T^2)$ case. On one hand, this is motivated by the increasingly important role played by integrable systems and higher spin symmetries in many areas of physics and mathematics. The best known examples are given by the Virasoro algebra, which underlies the physics of $2d$ conformal field theories (CFT) and its W_k -extensions. On the other hand, today it is well recognized that $2d$ conformal symmetry and its W_k higher spin extensions are intimately related to the algebra of $\text{diff}(S^1)$ and $\text{diff}(T^2)$, respectively [10–13]. In this context and after a setup of our conventional notations and basic definitions, we develop a systematic analysis leading to an explicit derivation of the KdV and Burgers differential equations. These systems are based on particular $\text{diff}(T^2)$ -non standard Lax operators $\{\log H, \cdot\}^2 + u_2$ and $\{\log H, \cdot\} + u_1$, respectively, where the Hamiltonian vector field $\xi_H \equiv \{\log H, \cdot\}$ plays the role of the derivation in $\text{diff}(T^2)$.

Among the results of this study is the possibility

to connect those systems, whose fields u_k are living on the bidimensional torus T^2 , through a consistent mapping that we will setup. Several important properties are discussed with some concluding remarks at the end.

2. Setup of the KdV Integrable Hierarchy

2.1. $\text{Diff}(T^2)$: Basic properties

In this section, we give a general setting of basic properties of the algebra of bianalytic fields defined on a bidimensional torus T^2 .

1) The two dimensional torus T^2 is viewed as a submanifold of the $4d$ real space $\mathbf{R}^4 \approx \mathbf{C}^2$ parametrized by two independent complex variables z and ω and their conjugates \bar{z} and $\bar{\omega}$ satisfying the constraint equation $z\bar{z} = \omega\bar{\omega} = 1$. Solutions of these equations are given by $z = e^{in\theta}, \omega = e^{im\psi}$, where n and m are two integers and where θ and ψ are two real parameters.

2) We identify the ring \mathcal{R} of bianalytic fields on T^2 with $\mathcal{R} \equiv \widehat{\Sigma}^{(0,0)}$ the tensor algebra of bianalytic fields of arbitrary conformal spin. This is a an infinite dimensional $SO(4)$ Lorentz representation that can be written as

$$\widehat{\Sigma}^{(0,0)} = \oplus_{k \in \mathbb{Z}} \widehat{\Sigma}_{(k,k)}^{(0,0)} \quad (1)$$

Where, the $\widehat{\Sigma}_{(k,k)}^{(0,0)}$'s are one dimensional $SO(4)$ irreducible modules corresponding to functions of bianalytic conformal spin (k, k) . The generators of $\widehat{\Sigma}_{(k,l)}^{(0,0)}$ are biperiodic arbitrary functions that we

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generally denote by $f(z, \omega)$ given by

$$f(z, \omega) = \sum_{n,m \in \mathbb{Z}} f_{nm} z^n \omega^m, \quad \partial_z f = \partial_{\bar{\omega}} f = 0 \quad (2)$$

Where, the constants f_{nm} are the Fourier modes of f . This is nothing but a generalization of the usual Laurent expansion of conformal fields on the complex plane \mathbb{C} . Note that the integers n and m carried by the Fourier modes f_{nm} are nothing but the $U(1) \times U(1)$ Cartan charges of the $SO(4) \approx SU(2) \times SU(2)$ Lorentz group of the Euclidean space \mathbb{R}^4 . Banalytic functions on \mathbb{C}^2 carrying $U(1) \times U(1)$ charges r and s and generalizing Eqn. (2) are given by

$$f_{(r,s)}(z, \omega) = \sum_{n,m \in \mathbb{Z}} f_{nm} z^{n-r} \omega^{m-s}; \quad r, s \in \mathbb{Z} \quad (3)$$

The coefficients f_{nm} are given by

$$f_{mn} = \oint_{c_1} \frac{dz}{2i\pi} z^{-n-l+r} \oint_{c_2} \frac{d\omega}{2i\pi} \omega^{-m+ -l+s} f_{(r,s)}(z, \omega) \quad (4)$$

Where, $c_1 \times c_2$ is the contour surrounding the singularity $(z, \omega) = (0, 0)$ in the complex space.

3) The special subset $\widehat{\Sigma}_{(k,k)}^{(0,0)} \subset \mathcal{R}$ is generated by banalytic functions $f_{(k,k)}$, $k \geq 2$. They

Objects \mathcal{O}

$$\begin{aligned} & z, \omega, \partial_z, \partial_{\bar{\omega}} \\ & L_{k,l} \\ & W_s(z, \omega), s = 2, 3, \dots \\ \{f, g\}^{(k)} &= \underbrace{\{f, \{f, \dots \{f, g\}\}\}}_k \\ \{f, g\}^k &= \underbrace{\{f, g\} \dots \{f, g\}}_k \\ \xi_H &= \underbrace{\{\log H, \cdot\}}_{Res} \end{aligned}$$

2.2. The space $\widehat{\Sigma}_n^{(r,s)}$ and conformal symmetry

To start, let us make clear that the space $\widehat{\Sigma}_n^{(r,s)}$ contains differential operators of fixed conformal spin (n, n) and degrees (r,s) , type

$$\mathcal{L}_n^{(r,s)}(u) = \sum_{i=r}^s u_{n-i}(z, \omega) \circ \xi_H^i \quad (8)$$

can be thought of as the higher spin currents involved in the construction of the W -algebra on T^2 [12]. As an example, the following fields

$$\begin{aligned} W_{(2,2)} &= u_{(2,2)}(z, \omega) \\ W_{(3,3)} &= u_{(3,3)}(z, \omega) - \frac{1}{2} \{\log H, u_{(2,2)}\} \end{aligned} \quad (5)$$

are shown to play the same role as that of the spin-2 and spin-3 conserved currents of the Zamolodchikov W_3 algebra [8, 9]. Next we will denote, for simplicity, the fields $u_{(k,k)}(z, \omega)$ of conformal spin (k, k) , $k \in \mathbb{Z}$ simply as $u_k(z, \omega)$.

4) The Poisson bracket on T^2 is defined as follows

$$\{f, g\} = \partial_z f \partial_{\bar{\omega}} g - \partial_z g \partial_{\bar{\omega}} f \quad (6)$$

with $\{z, \omega\} = 1$. We denote $\{f, \cdot\} = \xi_f$ and $\xi_f \cdot g = \{f, \cdot\} \cdot g = \{f, g\} + g\{f, \cdot\}$. Equivalently this shows how the Poisson bracket on the torus can play the role of a derivation. For convenience, we shall adopt the following notation, $\xi_H \equiv \xi_{\log H}$ has been the Hamiltonian vector field operator associated with the arbitrary function H .

5) We present here the essential properties of objects involved in this study.

The conformal weight $|\mathcal{O}|$

$$\begin{aligned} & |z| = (-1, 0), |\omega| = (0, -1), |\partial_z| = (1, 0), |\partial_{\bar{\omega}}| = (0, 1) \\ & |L_{k,l}| = (-k, -l) \\ & |W_s(z, \omega)| = (s, s) \\ \{f, g\}^{(k)} &= (k, k) + k|f| + |g| \\ \{f, g\}^k &= (k, k) + k|f| + k|g| \\ \xi_H &= (1, 1) \\ Res &= (1, 1) \end{aligned} \quad (7)$$

These are ξ_H 's polynomial differential operators extending the Hamiltonian field $\xi_H = \{\log H, \cdot\}$. Elements $\mathcal{L}_n^{(r,s)}(u)$ of $\widehat{\Sigma}_n^{(r,s)}$ are a generalization to T^2 of the well known KdV operator $\partial_z^2 + u_2(z)$. Moreover, Eqn. (8), which is well defined for $s \geq r \geq 0$, may be extended to negative integers by introducing pseudo-differential operators of the type ξ_H^{-k} ,

$k > 1$

$$\xi_H^n f(z, \omega) = \sum_{s=0}^n c_n^s \{ \log H, f \}^{(s)} \xi_H^{n-s} \quad (9)$$

and

$$\xi_H^{-n} f(z, \omega) = \sum_{s=0}^{\infty} (-)^s c_{n+s-1}^s \{ \log H, f \}^{(s)} \xi_H^{-n-s} \quad (10)$$

Where, the k^{th} -order derivative $\{ \log H, f \}^{(k)} = \underbrace{\{ \log H, \{ \log H, \dots \{ \log H, f \} \dots \}}}_{k \text{ times}}$ on the torus T^2 is

the analogue of $f^{(k)} = \frac{\partial^k f}{\partial z^k}$, the k th derivative of f in the standard case. The algebra $sl_n - \widehat{\Sigma}_n^{(0,n)}$ describes simply the coset space $\widehat{\Sigma}_n^{(0,n)} / \widehat{\Sigma}_n^{(1,1)}$ of sl_n -Lax operators on the torus T^2 given by

$$\mathcal{L}_n(u) = \xi_H^n + \sum_{i=0}^{n-2} u_{n-i} \xi_H^i \quad (11)$$

Where, we have set $u_0 = 1$ and $u_1 = 0$. This is a natural generalization of the well known differential sl_2 -Lax operator $\mathcal{L}_2 = \xi_H^2 + u_2$ associated to the KdV integrable hierarchy on the torus T^2 that we will discuss later. Consider the KdV Lax operator that we can write by virtue of the Miura transformation as

$$\begin{aligned} \mathcal{L}(u_2) &= \xi_H^2 + u_2(z, \omega) \\ &= (\xi_H + \{ \log H, \phi \}) \times (\xi_H - \{ \log H, \phi \}) \end{aligned} \quad (12)$$

Where, ϕ is a Lorentz scalar field. As a result we have

$$u_2 = -\{ \log H, \phi \}^{(2)} - \{ \log H, \phi \}^2 \quad (13)$$

describing the classical version of the stress energy momentum tensor of conformal field theory on the torus T^2 . Using bicomplex coordinates, we can write

$$\mathcal{T}(z, \omega) \equiv u_2(z, \omega) = -\{ \log H, \phi \}^{(2)} - \{ \log H, \phi \}^2 \quad (14)$$

The conservation for this bianalytic conformal current $\mathcal{T}(z, \omega)$, leads us to write the following differential equation

$$\{ \log \bar{K}, \{ \log H, \phi \} \} = e^{2\phi} \quad (15)$$

Where, $\bar{K} = K(\bar{z}, \bar{\omega})$ is an arbitrary bianalytic function of \bar{z} and $\bar{\omega}$ carrying in general an (\bar{n}_0, \bar{m}_0) $U(1) \times U(1)$ charge. Note also that \bar{K} is not necessarily the complex conjugate of the function H

considered earlier. Our experience with conformal field theory and integrable systems leads us to conclude that the later "second order" differential equation is nothing but the conformal Liouville like equation of motion on the torus T^2 . This equation of motion is known to appear in this context as a compatibility relation with the conservation of the stress energy momentum tensor $\mathcal{T}(z, \omega)$ namely

$$\{ \log \bar{K}, \mathcal{T}(z, \omega) \} = 0 \quad (16)$$

or equivalently

$$\begin{aligned} &\{ \log \bar{K}, \{ \log H, \phi \}^{(2)} \} \\ &+ 2\{ \log H, \phi \} \{ \log \bar{K}, \{ \log H, \phi \} \} = 0 \end{aligned} \quad (17)$$

2.3. The KdV equation on T^2

The KdV-like Lax operator

$$\mathcal{L}_{KdV} = \xi_H^2 + u_2(z, \omega) \quad (18)$$

belongs to the coset space $\widehat{\Sigma}_2^{(0,2)} / \widehat{\Sigma}_2^{(1,1)}$. As it is known from standard references in non-linear integrable models, we can set by analogy

$$\frac{\partial \mathcal{L}}{\partial t_{2n+1}} = [(\mathcal{L})_+^{\frac{2n+1}{2}}, \mathcal{L}] \quad (19)$$

which gives the $n - th$ evolution equation of the KdV-hierarchy. The index $+$ in Eqn. (19), stands for the local part of the pseudo-differential operator $\mathcal{L}^{\frac{2n+1}{2}}$ defined as follows $\mathcal{L}^{\frac{2n+1}{2}} = \mathcal{L}^{\frac{1}{2}} \circ \mathcal{L}^n$, where $\mathcal{L}^{\frac{1}{2}}$ is nothing but the half power of the KdV Lax operator. It describes a pseudo-differential operator weight $|\mathcal{L}^{\frac{1}{2}}| = (1, 1)$. The non linear pseudo-differential operator $\mathcal{L}^{\frac{2n+1}{2}}$ describes the $(2n + 1)^{th}$ power of $\mathcal{L}^{\frac{1}{2}}$

$$\begin{aligned} \mathcal{L}^{\frac{1}{2}} &= \xi_H + \frac{1}{2} u_2 \xi_H^{-1} - \frac{1}{4} \{ \log H, u_2 \} \xi_H^{-2} \\ &+ [\frac{1}{8} \{ \log H, u_2 \}^{(2)} - \frac{1}{8} u_2^2 \xi_H^{-3} \dots \end{aligned} \quad (20)$$

Where, the coefficients are explicitly determined by putting $\mathcal{L}_2 = (\mathcal{L}^{\frac{1}{2}} \circ \mathcal{L}^{\frac{1}{2}})$.

Consider special orders of the hierarchy Eqn. (19) parametrized by the index n . For $n = 0$ we get

$$\frac{\partial \mathcal{L}}{\partial t_1} = [(\mathcal{L})_+^{\frac{1}{2}}, \mathcal{L}] \quad (21)$$

where $(\mathcal{L})_+^{\frac{1}{2}} = \xi_H = \{ \log H, . \}$. We show also that Eqn. (21) corresponds simply to the chiral wave equation,

$$\frac{\partial u_2}{\partial t_1} = \{ \log H, u_2 \} \quad (22)$$

For $n = 1$, we have

$$\frac{\partial \mathcal{L}}{\partial t_3} = [(\mathcal{L})_+^{\frac{3}{2}}, \mathcal{L}] \tag{23}$$

where $(\mathcal{L})_+^{\frac{3}{2}}$ is explicitly given by

$$(\mathcal{L})_+^{\frac{3}{2}} = \xi_H^3 + \frac{3}{2}u_2\xi_H + \frac{3}{4}\{\log H, u_2\} \tag{24}$$

Injecting this expression into Eqn. (23) we get a non linear differential equation giving the evolution of the spin-2 conformal current u_2 , once some easy algebraic manipulations are performed. This is nothing but the KdV equation on the bidimensional torus T^2 given by

$$\frac{\partial u_2}{\partial t_3} = \frac{3}{2}u_2\{\log H, u_2\} + \frac{1}{4}\{\log H, u_2\}^3 \tag{25}$$

The same computations hold for the other evolution equations.

3. Conserved Quantities

Actually we know that the KdV equation in $\text{diff}(S^1)$ case is an integrable equation. This is because its non linear behavior dealing with solitonic solutions implies the existence of an infinite number of conserved quantities. The determination of these conserved quantities is well known in the standard case. We are presently looking for the $\text{diff}(T^2)$ extension and its impact on the integrability process. Let $\mathcal{Q}[u_i]$ be a conserved quantity for which we assume the following

$$\frac{d\mathcal{Q}[u_i]}{dt} = [\mathcal{Q}[u_i], \mathcal{H}] \tag{26}$$

where \mathcal{H} is the Hamiltonian of the system and $\mathcal{Q}(u_i)$ reads in terms of the charge-density as

$$\mathcal{Q}[u_i] = \int dz.d\omega\rho[u_i] \tag{27}$$

For the time-independent charges $\mathcal{Q}(u_i)$ we get the following continuity equation

$$\frac{\partial \rho[u_i]}{\partial t_k} + \{\log \tilde{H}, j[u_k]\} = 0 \tag{28}$$

Using the following natural property

$$\frac{1}{n}\{\log \tilde{H}, f^n\} = f^{n-1}\{\log \tilde{H}, f\} \tag{29}$$

the KdV equation (25) reads as

$$\begin{aligned} \frac{\partial u_2}{\partial t_k} &= \frac{3}{2}\{\log \tilde{H}, \frac{u^2}{2}\} + \frac{1}{4}\{\log \tilde{H}, u\}^3 \\ &= \{\log \tilde{H}, [\frac{3u^2}{4} + \frac{1}{4}\{\log \tilde{H}, u\}^2]\} \end{aligned} \tag{30}$$

When combined with the continuity equation (Eqn. (28)), it leads to

$$\begin{aligned} \rho_0[u] &= u_2 \\ j_0 &= \frac{3u^2}{4} + \frac{1}{4}\{\log \tilde{H}, u\}^2 \end{aligned} \tag{31}$$

Thus, a constant of motion can be extracted, namely

$$\begin{aligned} \mathcal{Q}_0 = \mathcal{H}_0 &= \int dz.d\omega\rho_0[u_i] \\ &= \int dz.d\omega u_2 \end{aligned} \tag{32}$$

The next step concerns the determination of other constant of motion. Consider once again the KdV equation (25), we obtain

$$\begin{aligned} \frac{\partial}{\partial t_k}(\frac{1}{2}u^2) &= \{\log \tilde{H}, [\frac{u^3}{3} - \frac{1}{8}\{\log \tilde{H}, u\}^2 \\ &\quad + \frac{1}{4}u\{\log \tilde{H}, u\}^2]\} \end{aligned} \tag{33}$$

Consequently, a second continuity equation can be extracted, that is

$$\frac{\partial \rho[u_i]}{\partial t_k} + \{\log \tilde{H}, j[u_k]\} = 0 \tag{34}$$

with

$$\begin{aligned} \rho_1[u] &= u_2 \\ j_1 &= -\frac{u^3}{3} + \frac{1}{8}\{\log \tilde{H}, u\}^2 - \frac{1}{4}\{\log \tilde{H}, u\}^2 \end{aligned} \tag{35}$$

The second constant of motion is then given by

$$\begin{aligned} \mathcal{Q}_1 = \mathcal{H}_1 &= \int dz.d\omega\rho_1[u_i] \\ &= \int dz.d\omega(\frac{1}{2}u^2) \end{aligned} \tag{36}$$

4. Lax-pair Representation

4.1. Lax-pair of the KdV equation

It is commonly known that Lax pair operators, once they exist, play a central role in proving the integrability. An integrable equation, which posses the the Lax representation, can be rewritten into the form of Lax equation given by

$$[\mathcal{L} = \xi_H^2 + u_2, \mathcal{P} + \partial_{t_{KdV}}] = 0 \tag{37}$$

with $\partial_t = \frac{\partial}{\partial t}$ and $t_{KdV} \equiv t_3$. Given a Lax operator \mathcal{L} , the crucial point in the Lax-pair technique is to find a corresponding operator \mathcal{P} constrained by Eqn. (37). This problem is very difficult to solve in general. However, putting some ansatz on \mathcal{P} , can help to get a wide class of solutions.

Ansatz: $\mathcal{P} = \xi_H^r \circ \mathcal{L}^s + \mathcal{P}'$

This ansatz reduces, in some sense, the problem for \mathcal{P} to that for \mathcal{P}' . So, let us consider the KdV Lax operator corresponding to $r = s = 1$, the bracket (Eqn. (37)) reduces, after easy computations, to

$$[\xi_H^2 + u_2, \mathcal{P}'] = \partial_{t_3} u_2 + u_2 \cdot \{ \log H, u_2 \} + \{ \log H, u_2 \} \xi_H^2 \tag{38}$$

Since the left side of this equation is a term that belongs to the ring \mathcal{R} , consistency requires that we should delete the term in ξ_H^2 appearing on its right side. This can be done if one takes the following form for the operator \mathcal{P}'

$$\mathcal{P}' = \frac{1}{2} u_2 \xi_H - \frac{1}{4} \{ \log H, u_2 \} \tag{39}$$

Plugging these expressions into the Lax equation (Eqn. (37)), one obtains an expression for the operator \mathcal{P} similar to that of $(\mathcal{L}^{\frac{3}{2}})_+$, in fact we get

$$\mathcal{P} = \xi_H^3 + \frac{3}{4} \{ \log H, u_2 \} + \frac{3}{2} u_2 \xi_H \tag{40}$$

With this expression of the operator \mathcal{P} , the Lax equation (Eqn. (39)) leads us to recover the same form of the KdV equation established in Eqn. (25). We have then an explicit form of the Lax-pair $(\mathcal{L}, \mathcal{P})$, associated with the KdV equation on torus T^2 .

4.1.1. Lax-pair of the Burgers equation

Our interest in the Burgers equation comes from the several important properties that are exhibited in the $\text{diff}(S^1)$ case. Let us recall for instance that in the standard pseudo-differential operator's formalism, this equation is associated with the following L -operator

$$L_{Burg} = \partial_x + u_1(x, t) \tag{41}$$

where the function u_1 is of conformal spin one. Note that in the language of complex numbers, where $z = x + it$ and $\bar{z} = x - it$, one can write $u = u(z, \bar{z})$ and show that under a conformal change of coordinates $z \rightarrow \tilde{z} = f(z)$ the u_1 currents transforms as an object of conformal spin 1. Now, we are ready to look for the $\text{diff}(T^2)$ version of the Burgers equation associated to

$$\mathcal{L}_{Burg} = \xi_H + u_1(z, \omega) \tag{42}$$

This is a local differential operator of the generalized n -KdV hierarchy's family ($n = 1$), that we can

interpret as a result of the truncation of a pseudo differential operator of KP-hierarchy type

$$\mathcal{L}_{KP} = \xi_H + u_1(z, \omega) + u_2(z, \omega) \circ \xi_H^{-1} + u_3(z, \omega) \circ \xi_H^{-2} + \dots \tag{43}$$

of the space $\widehat{\Sigma}_1^{(-\infty, 1)}$. The local truncation is simply given by

$$\begin{aligned} \widehat{\Sigma}_1^{(-\infty, 1)} &\rightarrow \widehat{\Sigma}_1^{(0, 1)} \equiv [\widehat{\Sigma}_1^{(-\infty, 1)}]_+ \\ &\equiv \widehat{\Sigma}_1^{(-\infty, 1)} / \widehat{\Sigma}_1^{(-\infty, -1)} \end{aligned} \tag{44}$$

or equivalently

$$\mathcal{L}_1(u_i) = \xi_H + \sum_{i=0}^{\infty} u_i \xi_H^{1-i} \rightarrow \xi_H + u_1 \equiv [\mathcal{L}_1(u_i)]_+ \tag{45}$$

The $\text{diff}(T^2)$ -Burgers equation is said to have the Lax representation if there exists a suitable pair of operators $(\mathcal{L}, \mathcal{P})$ so that the Lax commutation equation

$$[\mathcal{L}, \mathcal{P} + \partial_{t_{Burg}}] = 0 \tag{46}$$

with $t_{Burg} \equiv t_2$. To generate the Lax-pair associated to the $\text{diff}(T^2)$ -version of the Burgers equation one has to consider the following ansatz for the operator \mathcal{P}

$$\mathcal{P} = \xi_H \circ \mathcal{L} + \mathcal{P}' \tag{47}$$

or

$$\mathcal{P} = \xi_H^2 + u_1 \xi_H + \{ \log H, u_1 \} + \mathcal{P}' \tag{48}$$

Performing the straightforward computations, the $\text{diff}(T^2)$ Burgers Lax equation, reduces then to

$$\begin{aligned} [\mathcal{L}_{Burg} = \xi_H + u_1, \mathcal{P}'] &= \{ \log H, u_1 \} \circ \xi_H \\ &+ (u_1 \{ \log H, u_1 \} - \frac{1}{2} \{ \log H, u_1 \}^{(2)} + \partial_{t_{Burg}} u_1) \end{aligned} \tag{49}$$

Next, one have to take the following ansatz for the operator \mathcal{P}'

$$\mathcal{P}' = A \xi_H + B \tag{50}$$

Where, A and B are arbitrary functions for the moment. With this new ansatz for \mathcal{P}' , we have

$$\begin{aligned} [\xi_H + u_1, \mathcal{P}'] &= -(A \{ \log H, u_1 \} + \frac{1}{2} \{ \log H, A \}^{(2)} \\ &+ \frac{1}{2} [u, \{ \log H, A \}] + \{ \log H, B \} \\ &+ [u, B]) + (\{ \log H, A \} + [u_1, A]) \xi_H \end{aligned} \tag{51}$$

Identifying Eqns. (49) and (51) leads to the following constraints equations

$$\{logH, u_1\} = \{logH, A\} + [u_1, A] \quad (52)$$

and

$$\begin{aligned} (u_1 + A)\{logH, u_1\} + \partial_{t_{Burg}} u_1 &= \{logH, B\} \\ &+ [u_1, B] + \frac{1}{2}\{\{logH, A\}, u_1\} \\ &+ \frac{1}{2}\{\{logH, (u_1 - A)\}^{(2)}\} \end{aligned} \quad (53)$$

A natural solution of the first constraint equation (Eqn. (52)) is $A = u_1$. This implies a reduction of Eqn. (53) to

$$2u_1\{logH, u_1\} + \partial_{t_{Burg}} u_1 = \{logH, B\} + [u_1, B] \quad (54)$$

Since B is an object of conformal weight 2, We consider the following solution for Eqn. (53)

$$B = \alpha\{logH, u_1\} + \beta u^2 \quad (55)$$

with α and β are arbitrary coefficient numbers. Injecting this expression into Eqn. (53) gives the final expression of the $diff(T^2)$ -Burgers equation, namely

$$\partial_{t_{Burg}} u_1 + 2(1-\beta)u_1\{logH, u_1\} - \alpha\{\{logH, u_1\}^{(2)}\} = 0 \quad (56)$$

The associated Lax-pair is given by

$$\mathcal{L}_{Burg} = \xi_H + u_1 \quad (57)$$

and

$$\mathcal{P}_{Burg} = \xi_H^2 + 2u_1\xi_H + \beta u_1^2 + \alpha\{logH, u_1\} \quad (58)$$

4.2. Burgers-KdV mapping

This subsection will be devoted to another significant aspect of integrable models in $diff(T^2)$ framework. The principal focus, for the moment, is on the models discussed previously, namely the KdV and Burgers systems. Previously, we discussed the integrability of these two nonlinear systems and we noted that they are indeed integrable and this property is due to the existence of definite Lax pair operators $(\mathcal{L}, \mathcal{P})_{Burg}$ for each of the two models. Such existence implies the linearization of the models automatically. A crucial question that arises now is to know if there is a possibility to establish a mapping between the two systems. The idea to connect the two models

is originated from the fact that integrability for the KdV system is something natural due to conformal symmetry. We think that the strong backgrounds of conformal symmetry on T^2 can help to built integrability of the Burgers systems if one know how to establish such a connection.

Proposition 1:

Consider the Burgers operator $\mathcal{L}_{Burg}(u_1) = \xi_H + u_1 \in \widehat{\Sigma}_1^{(0,1)}$, for any given KdV operator $\mathcal{L}_{KdV}(u_2) = \xi_H^2 + u_2$ belongs to the space $\widehat{\Sigma}_2^{(0,2)}/\widehat{\Sigma}_2^{(1,1)}$, one can define the following mapping

$$\widehat{\Sigma}_1^{(0,1)} \hookrightarrow \widehat{\Sigma}_2^{(0,2)}/\widehat{\Sigma}_2^{(1,1)} \quad (59)$$

in such away that

$$\mathcal{L}_{Burg}(u_1) \mapsto \mathcal{L}_{KdV}(u_2) \equiv \mathcal{L}_{Burg}(u_1) \otimes \mathcal{L}_{Burg}(-u_1) \quad (60)$$

What we are assuming in this proposition is a strong constraint leading to connect the two spaces. This constraint is also equivalent to set

$$\widehat{\Sigma}_2^{(0,2)}/\widehat{\Sigma}_2^{(1,1)} \equiv \widehat{\Sigma}_1^{(0,1)} \otimes \widehat{\Sigma}_1^{(0,1)} \quad (61)$$

Next, we are interested in exploring the crucial key behind the previous Proposition. We then underline that this mapping is easy to highlight through the Miura transformation

$$\mathcal{L}_{KdV} = \xi_H^2 + u_2 = (\xi_H + u_1) \circ (\xi_H - u_1) \quad (62)$$

giving rise to

$$u_2 = -u_1^2 - \{logH, u_1\} \quad (63)$$

Proposition 1 can have a complete and consistent significance only if one manages to establish a connection between the differential equations associated with the two systems. At this stage, note that besides the principal difference due to conformal spin, we stress that the two nonlinear evolutions equations of KdV

$$\partial_{t_3} u_2 = \frac{3}{2}u_2\{logH, u_2\} + \frac{1}{4}\{\{logH, u_2\}^{(3)}\} \quad (64)$$

and of Burgers

$$\partial_{t_{Burg}} u_1 + 2(1-\beta)u_1\{logH, u_1\} - \alpha\{\{logH, u_1\}^{(2)}\} = 0 \quad (65)$$

are distinct by a remarkable fact, that is, the KdV flow $t_{KdV} \equiv t_3$ and the Burgers one $t_{Burg} \equiv t_2$ have different conformal weights: $[t_{KdV}] = (-3, -3)$, whereas $[t_{Burg}] = (-2, -2)$.

Now, we are constrained to circumvent the effect

of proper aspects specific to both the equations and consider the following second property:

Proposition 2:

By virtue of the Burgers-KdV mapping and dimensional arguments, the associated flows are related through the following ansatz

$$(\partial_{t_{Burg.}} \bullet) \leftrightarrow (\partial_{t_{KdV}} \bullet) \equiv \partial_{t_{Burg.}} \{ \log K, \bullet \} + \eta \{ \log K, \bullet \}^{(3)} \quad (66)$$

acting on arbitrary function \mathcal{F} in the following way

$$\partial_{t_{Burg.}} \mathcal{F} \leftrightarrow \partial_{t_{KdV}} \mathcal{F} \equiv \{ \log K, (\partial_{t_{Burg.}} \mathcal{F}) \} + \eta \{ \log K, \mathcal{F} \}^{(3)} \quad (67)$$

for an arbitrary parameter α . With respect to the assumption Eqn. (66), relating the two evolution

$$\begin{aligned} \partial_{t_3} u_2 &= 3u_1^3 \{ \log H, u_1 \} + 3 \{ \log H, u_1 \}^2 u_1 + \frac{3}{2} \{ \log H, u_1 \}^{(2)} u_1^2 - \frac{1}{2} \{ \log H, u_2 \}^{(3)} u_1 \\ &\quad - \frac{1}{4} \{ \log H, u_1 \}^{(4)} \\ &= -2a \{ \log H, u_1 \}^2 u_1 - 3a \{ \log H, u_1 \}^{(2)} u_1' - 2a \{ \log H, u_1 \}^2 \{ \log H, u_1 \}^{(2)} \\ &\quad - (b + \eta) \{ \log H, u_1 \}^{(3)} - 2(\eta + \frac{a}{2} + b) u_1 \{ \log H, u_1 \}^{(3)} \\ &= -4a \{ \log H, u_1 \}^2 u_1 - 2(b + 3\eta + \frac{3a}{2}) \{ \log H, u_1 \}^{(2)} \{ \log H, u_1 \} \\ &\quad - 2(b + \eta + \frac{a}{2}) \{ \log H, u_1 \}^{(3)} u_1 - 2a \{ \log H, u_1 \}^{(2)} u_1^2 - (b + \eta) \{ \log H, u_1 \}^{(4)} \end{aligned} \quad (69)$$

Once simplified, these expressions lead to a strong constraint on the Burgers equation. Performing a straightforward computation, one shows that

$$\{ \log H, u_1 \}^{(k)} \sim u_1^{k+1}, \quad 1 \leq k \leq 4 \quad (70)$$

which means that $\{ \log H, u_1 \} \sim u_1^2$, $\{ \log H, u_1 \}^{(2)} \sim u_1^3$ and so on.

Putting these constraint equations into the Burgers equation (65) one obtain the following differential equation

$$\partial_{t_{Burg.}} u_1 \sim \{ \log H, u_1 \}^{(2)} \quad (71)$$

This is an impressive result since the mapping between the flow of KdV and Burgers nonlinear differential equations affect the Burgers equation as follows

$$\begin{aligned} \partial_{t_{Burg.}} u_1 &\sim (\dots) u_1 \{ \log H, u_1 \} + (\dots) \{ \log H, u_1 \}^{(2)} \\ &\quad \leftrightarrow \partial_{t_{Burg.}} u_1 \sim \{ \log H, u_1 \}^{(2)} \end{aligned} \quad (72)$$

derivatives $\partial_{t_{Burg.}}$ and $\partial_{t_{KdV}}$, one should expect some strong constraint on the Burgers differential equation. Using *Proposition 2*, we have to identify the following three differential equations

$$\begin{aligned} \partial_{t_3} u_2 &= \frac{3}{2} u_2 \{ \log H, u_2 \} + \frac{1}{4} \{ \log H, u_2 \}^{(3)} \\ &= -2u_1 \partial_{t_3} u_1 - \partial_{t_3} \{ \log H, u_1 \}, \\ &= \partial_{t_2} \{ \log H, u_2 \} + \eta \{ \log H, u_2 \}^{(3)}. \end{aligned} \quad (68)$$

As a matter of simplicity, setting the Burgers equation as $\partial_{t_2} u_1 = a u_1 \{ \log H, u_1 \} + b \{ \log H, u_1 \}^{(2)}$ with $a = 2(\beta - 1)$ and $b = \alpha$ and performing the explicit computation, arising from the identification of the previous system of equations, we find

This is also equivalent to the argument that the proposed mapping induces a cancelation of the non linear term $\sim (\dots) u_1 \{ \log H, u_1 \}$ responsible of solitonic character at the level of the Burgers equation. We guess that a hidden extended $2d$ -conformal symmetry is behind the linearizability property induced by the Burgers-KdV mapping. This is because the conformal symmetry in the framework of KdV hierarchy is related in general to the sl_n -symmetry. In fact, we have to remark that the Burgers u_1 -current issued from the Miura like equation can be identified with the Liouville Lorentz scalar field ϕ as follows $u_1 \equiv \{ \log K, \phi \}$ describing the derivative of the Liouville Lorentz scalar field, while the KdV potential u_2 satisfying Eqn. (64) can then be identified with the conformal current \mathcal{T} given by Eqn. (14).

5. Concluding Remarks

We presented in this paper some important aspects of integrable KdV hierarchies dealing with higher

conformal spin symmetries on the bidimensional torus T^2 . These symmetries, generalizing the Frappat et al. conformal symmetries by adding currents of conformal spin $(3, 3)$ in a non standard way, are also shown to be derived, in their semi-classical form, from the GD bracket [13].

Note that KdV hierarchies on $\text{diff}(T^2)$ exhibit many remarkable features [14]. The first one concerns the introduction of new kind of derivatives taking the following form $\xi_H \equiv \{\log H, \cdot\} = \partial_z \log H \partial_\omega - \partial_\omega \log H \partial_z$ for arbitrary bianalytic function $H(z, \omega)$. Besides the above established results, we tried also to understand furthermore the meaning of integrability of nonlinear systems on T^2 . The principal focus was on the KdV and Burgers systems. A first step was to derive these two equations using the above systematic algebraic formulation in the context of Lax-pair building program. Concerning the derived KdV system, this is an integrable model due to the existence of a Lax pair operators $(\mathcal{L}_{KdV}, \mathcal{P}_{KdV})$. This existence is an important indication of integrability, but we guess that the realistic source of integrability of this model is the underlying conformal symmetry. For the Burgers system [15], to check its integra-

bility we proceeded to an explicit derivation of the Lax pair operators $(\mathcal{L}_{Burg}, \mathcal{P}_{Burg})$ giving rise to the following differential equation

$$2u_1 \{\log H, u_1\} + \partial_{t_{Burg}} u_1 = \{\log H, B\} + [u_1, B]. \quad (73)$$

Solving this equation, we get the explicit form of the requested Lax operator.

Concerning the possibility to establish a correspondence between the KdV and the Burgers systems, we succeeded in fact to build a mapping from the Burgers system to the KdV one. The main line of this mapping deals with the following ansatz

$$\partial_{t_{Burg}} \mathcal{F} \leftrightarrow \partial_{t_{KdV}} \mathcal{F} \equiv \{\log K, (\partial_{t_{Burg}} \mathcal{F})\} + \eta \{\log K, \mathcal{F}\}^{(3)} \quad (74)$$

for an arbitrary parameter η .

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