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Conformally Flat Cylindrically Symmetric Perfect Fluid Distribution in Einstein-Cartan Theory

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We have studied a static conformally flat cylindrical symmetric perfect fluid distribution with improved energy-momentum tensor developed by Ray and Smalley and obtain an exact solution in the context of Einstein-Cartan Theory. The explicit expressions for pressure, spin, energy density, expansion, rotation and shear have also been found.

1. Introduction

There are several solutions with different symmetries in various theories of gravitation; conformally flat solutions are among them. However, conformally flat solutions take a special status in all theories of gravitation as the number of unknown functions are reduced and are also helpful to interpret the surplus rotation velocity of the celestial objects far from the center of galaxies. Moreover, these solutions are very useful for cosmology. For example, in an exact Friedmann–Robertson–Walker (FRW) universe, the Weyl tensor always vanishes. Therefore, an arbitrary scale factor is a solution of a void universe.

In the context of the general theory of relativity (GTR), several authors have studied various static and non-static conformally flat solutions. It was Buchdahl [1] who studied a static conformally flat Schwarzschild interior solution, and then Burman [2] has studied the geodesic of the particles in conformally flat-space-time. Singh and Abdussattar [3] have studied a non-static conformally flat Schwarzschild interior solution and Roy and Bali [4] have studied a non-static conformally flat spherically symmetric space-time. Pandey and Tiwari [5] have studied a conformally flat spherically symmetric charged perfect fluid distribution. In [6], van den Berg has studied perfect fluid solutions performing a conformal transformation on the Schwarzschild exterior space-time. Moreover, Hansraj [7] constructed a perfect fluid by performing a conformal transformation on a known non-conformally flat vacuum Schwarzschild exterior solution and shown that all perfect fluid space-times conformal to the exterior Schwarzschild line element are necessarily static. Hansraj et al. [8] also studied expanding, shearing and accelerating isotropic plane-symmetric universe with conformal Kasner geometry.

In the context of Einstein-Cartan Theory (ECT) [9–15], Bedran and Som [16] studied a conformally flat solution of a static dust sphere. Kalyanshetti and Waghmode [17] considered the static conformally flat solutions for spherically symmetric perfect fluid distribution. Katkar and Patil also studied a static [18] and a non-static [19] conformally flat spherically symmetric solutions.

Although cylindrically symmetric solutions are widely investigated in the frame-work of ECT, for example, Prasanna [20], Soleng [21], Tsoubelis [22] and Manna et. al. [23] studied some exact solutions for cylindrically symmetric perfect fluid distributions. But, the conformally flat cylindrically symmetric solutions are seldom studied. Hence, here we want to study a conformally flat cylindrically symmetric solution in the context of ECT. In this paper, we study a static conformally flat cylindrically symmetric perfect fluid distribution with improved energy-momentum developed by Ray and Smalley [24]. The structure of this paper is as follows: In Sec.

2, we discuss metric and curvature. In Sec. 3, we set up the field equations and obtain their exact solutions. Finally, we present our conclusion in Sec. 4.

2. Metric and Curvature

The Cartan structures of equations in Einstein-Cartan manifold [25] are

$$d\theta^i + \omega^i{}_j \wedge \theta^j = \frac{1}{2} Q^i{}_{jk} \theta^i \wedge \theta^k \quad (1)$$

$$\Omega^i{}_j = d\,\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j \quad (2)$$

$$Q^i{}_{jk} - \delta^i_j Q^l{}_{lk} - \delta^i{}_k Q^l{}_{jl} = k\, S^i{}_{jk}\,, \quad (3)$$

Where, $k = \frac{8\pi G}{C^4}$ is coupling constant and $S^i{}_{jk}$ are the independent components of spin tensor. The classical description of the spin tensor is defined by the relation $S^i{}_{jk} = S_{jk} U^i$, where S_{jk} is tensor of the density of spin and U^i is the four velocity of fluid which obeys the Weyssenhoff condition [26],

$$S_{ij}{}^j = S_{ij} U^j = 0. \quad (4)$$

A static, conformally flat cylindrical-symmetric perfect fluid is considered to be given by the space-time metric

$$ds^2 = e^{2\lambda}[-e^{2\nu}dt^2 + e^{2(\mu-\nu)}(dr^2 + dz^2) + r^2 e^{-2\nu} d\phi^2], \quad (5)$$

Where, λ, conformal factor is function of r only and metric functions μ, ν are also function of r only.
The orthonormal tetrads of given metric are given by

$$\theta^0 = e^{\lambda+\nu}dt, \quad \theta^1 = e^{\lambda+\mu-\nu}dr, \quad \theta^2 = re^{\lambda-\nu}\, d\phi, \theta^3 = e^{\lambda+\mu-\nu}dz. \quad (6)$$

So the tetrads form of the stationary static cylindrically symmetric metric can be written as

$$ds^2 = g_{ij}\theta^i\,\theta^j, \quad (6.a)$$

along with the metric tensor

$$g_{ij} = diagonal\,(-1,+1,+1,+1).$$

We consider the spin of individual particles are aligned along the symmetry axis (z axis) in a comoving frame, so that the only non-vanishing components are

$$S_{ij} = S_{12} = -S_{21} = 2S. \quad (7)$$

Now using Cartan structure of equations (1-3) along with (6), (6.a) and (7), we get the components of ω_{ij}. Using the value of ω_{ij} we can obtain curvature two-form $\Omega^i{}_j$ and hence Ricci tensors and Scalar curvature. Here, the non-zero components of Ricci tensor taking $\kappa = 1$ are as follows:

$$R_{00} = 2S^2 - (2\lambda'\nu' + \nu'' + \frac{\nu'}{r} + \lambda'' + 2\lambda'^2 + \frac{\lambda'}{r})e^{2(\nu-\lambda-\mu)} \quad (8)$$

$$R_{11} = (2\nu'^2 + 2\lambda'\nu' + 3\nu'' + \frac{\nu'}{r} + \frac{\mu'}{r} - \lambda'' - \mu'' - \frac{\lambda'}{r})e^{2(\nu-\lambda-\mu)} \quad (9)$$

$$R_{22} = (\nu'' + 2\nu'\lambda' - 2\nu'^2 + \frac{2\nu'}{r} - \lambda'' - \frac{\lambda'}{r})e^{2(\nu-\lambda-\mu)} \quad (10)$$

$$R_{33} = e^{2(\nu-\lambda-\mu)}(\nu'' - 2\nu'^2 + 2\nu'\mu' + 2\nu'\lambda' + \frac{\nu'}{r} - \frac{\mu'}{r} - \lambda'' - \mu'' - \frac{\lambda'}{r}) \quad (11)$$

$$R_{02} = -(S' - S\nu' + S\mu' + S\lambda')e^{\nu-\mu-\lambda}. \quad (12)$$

$$R_{20} = -(S' + S\nu' + S\mu' + S\lambda')e^{\nu-\mu-\lambda}. \quad (13)$$

Here, equations (12) and (13) give $\nu = 0$ or constant. We get the expression for spin as follows:

$$S = S_0 e^{-(\mu+\lambda)}, \quad (14)$$

Where, S_0 is constant. Now we have the Ricci's scalar curvature as follows:

$$R = -2S^2 + 2(\lambda'^2 - \lambda'' - \frac{\lambda'}{r} - \mu'')e^{-2(\lambda+\mu)}. \quad (15)$$

3. Field Equations and Exact Solution

We assume the material distributions to be that of a perfect fluid constituted of spin particles which are Weyssenhoff type. Here, we have considered Ray and Smalley [24] energy-momentum tensor for the perfect fluid and is given by

$$t_{ij}^{RS} = \{(1+\epsilon)\rho + P\}U_iU_j + Pg_{ij} + U_{(j}\, S_{i)k}\dot{U}^k - \bar{\omega}_{(i\}}^k S_{j)k} + U_{(i}S^k{}_{j)}\bar{\omega}_{kl}U^l, \qquad (16)$$

Where, dot denotes the covariant derivatives with respect to U^i and $(1+\epsilon)\rho$ is the energy density, P is the pressure and $\bar{\omega}_{ij}$ is the spin angular velocity of the fluid. For perfect fluid using the corrected energy-momentum tensors [24] are

$$t_{ij}^{RS} = diagonal(m, P_1, P_2, P_3), \qquad (17)$$

Where,

$$P_1 = P - 2S^2, \qquad P_2 = P - 2S^2, \qquad P_3 = P \quad and \quad m = (1+\epsilon)\rho\,. \qquad (18)$$

$$S^2 - (2\lambda'' + \lambda'^2 + \frac{2\lambda}{r} + \mu'')e^{-2(\lambda+\mu)} = m. \qquad (19)$$

$$S^2 + \left(\frac{\mu'}{r} - \lambda'^2\right)e^{-2(\lambda+\mu)} = P - 2S^2. \qquad (20)$$

$$S^2 + (\mu'' - \lambda'^2)e^{-2(\lambda+\mu)} = P - 2S^2. \qquad (21)$$

$$S^2 + \left(-\frac{\mu'}{r} - \lambda'^2\right)e^{-2(\lambda+\mu)} = P. \qquad (22)$$

Hence from equations (20) and (21) we get,

$$\mu'' = \frac{\mu'}{r}, \qquad (23)$$

and from equations (14) and (20-22) we get,

$$\mu = \mu_0 r^2 = -\frac{(S_0 r)^2}{2}. \qquad (24)$$

Now, for the conformal flatness, tetrad components of Weyl tensor are zero. Hence, we get,

$$R_{abcd} = \frac{1}{2}(R_{bc}\eta_{ad} - R_{bd}\eta_{ac} - R_{ac}\eta_{db} + R_{ad}\eta_{bc}) - \frac{1}{6}R(\eta_{bc}\eta_{da} - \eta_{db}\eta_{ac}). \qquad (25)$$

Imposing the condition of conformally flat of equation (25) we get,

$$\lambda = ln\frac{1}{1 + S_0^2\, r^2}. \qquad (26)$$

The expression for the pressure, spin and energy density are as follows:

$$P = S_0^2(1 - S_0 r^2)^2 e^{S_0^2 r^2},$$

$$S^2 = S_0^2(1 + S_0^2 r^2)^2 e^{S_0^2 r^2},$$

$$m = 4S_0^2(2 - 3S_0^2 r^2)e^{S_0^2 r^2}. \qquad (27)$$

Using equations (24) and (26) we get the conformally flat line element as follows:

$$ds^2 = \frac{1}{1 + S_0^2 r^2}\left[dt^2 + e^{-S_0^2 r^2}(dr^2 + dz^2) + r^2 d\phi^2\right]. \qquad (28)$$

We know the expansion (Θ) monitors the convergence/divergence of the worldliness tangent to 4-velocity field, while the shear (σ_{ij}) and the vorticity (ω_{ij}) tensors describe kinematic anisotropies and the rotational behaviour of the field, respectively, and 4-acceleration $(\dot{U}_i)$ vector implies that the aforementioned worldliness are autoparallel curves or not. Here, the kinematical parameters like expansion, shear, vorticity and acceleration, taking $U_iU^i = -1$ are

$$\Theta = 0,$$
$$\sigma_{ij} = 0,$$
$$\omega_{12} = -\omega_{21} = S_0\,(1 + S_0^2 r^2)e^{S_0\, r},$$
$$\dot{U}_i = 0. \qquad (29)$$

4. Conclusions

In this study, we have obtained static, cylindrically symmetric, conformally flat interior solutions in the context of ECT using corrected energy-momentum tensor for perfect fluid developed by Ray and Smalley [24]. From equation (29), we have obtained a solution that is expansion free and shear free but rotating with zero acceleration. Hence, the motion of particles inside the cylinder will be geodesic.

Also, we observed that the spin of the gravitating matter produces a repulsive effect on pressure and influences the energy density, and also the geometry of space-time. We hope that our solution will be useful to study of rotating back holes, to study of FRW like models and for future study of cosmology.

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