Bound-state Solutions of Klein-Gordon and Schrödinger Equations for Arbitrary *l*-state with Linear Combination of Hulthén and Kratzer Potentials

Abdelmadjid Maireche

Laboratory of Physical and Chemical materials, Physics Department, Sciences Faculty, University of M'sila, Algeria
Email: abdelmadjid.maireche@univ-msila.dz

We present approximate solutions of the both the modified Klein–Gordon (MKGE) and modified Schrödinger equation (MSE) containing the modified Hulthén and modified Kratzer potential using the procedure of Bopp's shift method and perturbation theory in addition to the Greene–Aldrich approximation method of handling centrifugal barriers. This study is conducted in the relativistic and nonrelativistic non-commutative 3-dimensional real space (RNC: 3D-RS) and (NRNC: 3D-RS) symmetries, respectively. The Hulthén–Kratzer potential model is extended to include new radial terms. Furthermore, this potential model is proposed to study some selected diatomic molecules, namely N_2, I_2, CO, NO and HCl. The ordinary Bopp's shift method and perturbation theory are surveyed to get generalized excited states energy as a function of the shift energy and the energy E_{nl} of the HKP model. Furthermore, the obtained perturbative solutions of the discrete spectrum were dependent on Gamma function, the discreet atomic quantum numbers (j,l,s,m) and the potential parameters (V_0, α, r_e, D_e), and the NC-parameters, which are generated with the effect of (space-space) non-commutative properties. We have also applied our results on diatomic-molecules with spin-0 and spin-1, and have shown that the modified Klein-Gordon equation MKG under the MHKP model becomes similar to the Duffin–Kemmer equation.

1. Introduction

It is well recognized that the Hulthén potential [1] plays an essential role in several fields. For example, it is used to study the optical properties of quantum dots [2] , cosmic strings in the relativistic scales [3] and it has been applied to sub-atomic and atomic scales, such as nuclear and particle physics, atomic physics, condensed matter and chemical physics [4,5]. Furthermore, it is one of the important shortrange potentials in physics, which finds applications in a wide range of physical systems [5, 6]. In addition to that, it characterizes two important features, at a short distance; its behavior becomes identical to the screened coulomb potential, while for large distance, it becomes a decreasing exponential potential [4]. It is worth noting that the Kratzer-like potential can be used to describe the atomic, molecular physics, vibrational and rotational spectroscopy [7].

Currently, researchers became more interested in the state of combination between than two potentials or more than two potentials, such as a combination between the modified Kratzer potential plus screened Coulomb potential [8] and between Hellmann and Kratzer potential model [9]. We have studied the solutions of the modified Schrodinger equation with generalized Hellmann–Kratzer potential model in the symmetries of NRNCQM[10].

In 2019 H., Louis *et al.* studied the K-state solutions to the Dirac equation for the quadratic exponential-type potential plus Eckart potential and Coulomb-like tensor interaction using Nikiforov-Uvarov method [11]. We have studied the Klein–Gordon equation with modified Coulomb potential plus inverse-square root potential and the modified Coulomb plus inverse-square potential in the non-commutative 3-dimensional space [12-13].

Here we present a new model to describe Heavy-Light Mesons in the extended non-relativistic quark model under a new modified potential containing Cornell, Gaussian and inverse square terms in the symmetries of NCQM [14]. Very recently, J. A., Obu *et al.* [6] applied the Hulthén–Kratzer potential model to the study of the diatomic molecules N_2, I_2, CO, NO and HCl. In this work, we are motivated by many recent studies, such as the non-renormalizable electroweak interaction, quantum gravity, string theory, the noncommutative relativistic and nonrelativistic quantum mechanics that has attracted much attention of physical researchers [15-21]. The noncommutativity in space-time is not a new idea, it was first proposed by W. Heisenberg in 1930 and then it was developed by H. Snyder in 1947. Currently, there are several studies concerning the search for solutions to the various three basic equations in the relativistic and nonrelativistic state.

This work focuses on applying principle and the foundations of non-commutative theory [22-32]. The main objective to this work is to develop the work done by J. A., Obu, *et al.* and expanding in the symmetries of NCRQM and NCNRQM for the purpose to get more investigation in the microscopic scales and from achieving more scientific knowledge of elementary particles in the field of nanotechnology. The relativistic and nonrelativistic energy levels under modified Hulthén–Kratzer potential model have not been obtained yet in the context of the

NCRQM and NCNRQM. Furthermore, we hope to find new applications and profound physical interpretations using a new, updated model of the modified Hulthén–Kratzer potential, which has the following form:

$$\begin{aligned} V_{hmk}(r) &= D_e\left(\frac{r-r_e}{r_e}\right)^2 - \frac{V_0 e^{-\alpha r}}{1-e^{-\alpha r}} \rightarrow \\ V_{hmk}(\hat{r}) &\equiv V_{hmk}(r) - \frac{\partial V_{hmk}(r)}{\partial r}\frac{\vec{\mathbf{L}}\,\vec{\Theta}}{2r} + O(\Theta^2) \end{aligned} \quad (1)$$

The potential parameters will be defined in the next section. The new structure of RNCQM and NRNCQM based to new covariant non-commutative canonical commutations relations CNCCRs in Schrödinger, Heisenberg and Interactions pictures (SP, HP and IP), respectively, as follows [33-42]:

$$\begin{cases} \left[\hat{x}_\mu \overset{*}{,} \hat{p}_\nu\right] = \left[\hat{x}_\mu(t) \overset{*}{,} \hat{p}_\nu(t)\right] = \left[\hat{x}^I_\mu(t) \overset{*}{,} \hat{p}^I_\nu(t)\right] = i\hbar_{eff}\delta_{\mu\nu} \\ \left[\hat{x}_\mu \overset{*}{,} \hat{x}_\nu\right] = \left[\hat{x}_\mu(t) \overset{*}{,} \hat{x}_\nu(t)\right] = \left[\hat{x}^I_\mu(t) \overset{*}{,} \hat{x}^I_\nu(t)\right] = i\theta_{\mu\nu} \end{cases} \quad (2)$$

We have generalized the CNCCRs to include HP and IP. It should be noted that, in our calculation, we have used the natural units $c = \hbar = 1$. Here $\hbar_{eff}$ is the effective Planck constant, $\theta^{\mu\nu} = \varepsilon^{\mu\nu}\theta$ (θ is the non-commutative parameter), which are infinitesimals parameter if compared to the energy values and elements of antisymmetric 3 × 3 real matrix and $\delta_{\mu\nu}$ is the identity matrix. The symbol $(*)$ denote to the Weyl Moyal star product, which is generalized between two ordinary functions $f(x)g(x)$ to the new modified form $\hat{f}(\hat{x})\hat{g}(\hat{x}) \equiv f(x)*g(x)$ in the symmetries of (RNC: 3D-RS) and (NRNC: 3D-RS) as follows [43-50]:

$$\begin{aligned} (fg)(x) &\rightarrow (f*g)(x) = \exp\left(i\theta\varepsilon^{\mu\nu}\partial_{x_\mu}\partial_{x_\nu}\right) f(x_\mu) g(x_\nu) \\ &\cong fg(x) - \frac{i\varepsilon^{\mu\nu}}{2}\theta\partial^x_\mu f\,\partial^x_\nu g\Big|_{x_\mu = x_\nu} + O(\theta^2) \end{aligned} \quad (3)$$

The indices are $(\mu, \nu \equiv \overline{1,3})$, while $O(\theta^2)$ stands for second and higher-order terms of the non-commutative parameter. Physically, the term ($-\frac{i\varepsilon^{\mu\nu}}{2}\theta\partial^x_\mu f\,\partial^x_\nu g\Big|_{x_\mu = x_\nu}$) in the Eqn. (3) presents the effects of space-space non-commutative properties.

Furthermore, the new unified two operators $\hat{\xi}^H_\mu(t) = (\hat{x}_\mu \text{or } \hat{p}_\mu)(t)$ and $\hat{\xi}^I_\mu(t) = (\hat{x}^I_\mu \text{or } \hat{p}^I_\mu)(t)$ in HP and IP are depending on the corresponding new operators $\hat{\xi}^H_\mu \equiv \hat{x}_\mu \text{or } \hat{p}_\mu$ in SP from the following projections relations, respectively:

$$\begin{cases} \xi^H_\mu(t) = \exp\left(i\hat{H}^{hmk}_r(t-t_0)\right)\xi^S_\mu \exp\left(-i\hat{H}^{hmk}_r(t-t_0)\right) \\ \xi^I_\mu(t) = \exp\left(i\hat{H}^{hmk}_{or}(t-t_0)\right)\xi^S_\mu \exp\left(-i\hat{H}^{hmk}_{or}(t-t_0)\right) \end{cases} \Rightarrow$$

$$\begin{cases} \hat{\xi}^H_\mu(t) = \exp\left(i\hat{H}^{hmk}_{nc-r}(t-t_0)\right) * \hat{\xi}^S_\mu * \exp\left(-i\hat{H}^{hmk}_{nc-r}(t-t_0)\right) \\ \hat{\xi}^I_\mu(t) = \exp\left(i\hat{H}^{hmk}_{nc-or}(t-t_0)\right) * \hat{\xi}^S_\mu * \exp\left(-i\hat{H}^{hmk}_{nc-or}(t-t_0)\right) \end{cases} \quad (4)$$

Where the three unified coordinates $\xi^S_\mu \equiv (x_\mu \text{ or } p_\mu)$, $\xi^H_\mu(t) \equiv (x_\mu \text{or } p_\mu)(t)$ and $\xi^I_\mu(t) \equiv (x^I_\mu \text{or } p^I_\mu)(t)$ are represented in three relativistic quantum mechanics pictures, whereas the dynamics of new systems $\frac{\mathrm{d}\hat{\xi}_H(t)}{\mathrm{dt}}$ is described by the following equation of motion in the modified Heisenberg picture, as follows:

$$\begin{aligned} \frac{\mathrm{d}\xi^H_\mu(t)}{\mathrm{dt}} &= \left[\xi^H_\mu(t), \hat{H}^{hmk}_r\right] + \frac{\partial \xi^H_\mu(t)}{\partial t} \Rightarrow \\ \frac{\mathrm{d}\hat{\xi}_H(t)}{\mathrm{dt}} &= \left[\hat{\xi}^H_\mu(t) \overset{*}{,} \hat{H}^{hmk}_{nc-r}\right] + \frac{\partial \hat{\xi}^H_\mu(t)}{\partial t} \end{aligned} \quad (5)$$

The operators $\hat{H}^{hmk}_{or}$ and $\hat{H}^{hmk}_r$ are the free and global Hamiltonian for Hulthén–Kratzer potential model while $\hat{H}^{hmk}_{nc-or}$ and $\hat{H}^{hmk}_{nc-r}$ the corresponding Hamiltonians for MHKP model.

The present investigation aims at constructing a relativistic and non-relativistic non-commutative effective scheme for the modified Hulthén–Kratzer potential model. The rest of this manuscript is organized as follows: In the next section, we briefly review the Klein-Gordon equation with Hulthén–Kratzer potential model based on Ref. [6]. Section 3 is devoted to the study of modified Klein-Gordon equation MKGE by applying the ordinary Bopp's shift method and to obtain the effective potential of MHKP model. We find the expectation values of the radial terms, $1/r$, $1/r^3$ and $1/r^4$. Section 4 is devoted to obtaining the results and a discussion of the energy shift for the generalized n^{th} excited states, which is produced by the effects of perturbed spin-orbital and the generated new Zeeman interactions in the RNCQM. Then, we determine the energy spectra of diatomic molecules N_2, I_2, CO, NO, and HCl under MHKP model in the RNCQM symmetries. After that, we discuss the non-relativistic limits. The final section will be devoted to results and conclusions.

2. Revised Bound-state Solutions of Klein-Gordon Equation for Arbitrary l-state with Linear Combination of Hulthén and Kratzer Potentials in RQM

As already mentioned, our objective is to obtain the spectrum of modified Klein-Gordon equation with a modified Hulthén–Kratzer potential model in (RNC: 3D-RSP) and (NC: 3D-RSP) symmetries, we need to revise the corresponding Hulthén–Kratzer potential model in symmetries of ordinary relativistic quantum mechanics RQM [6]

$$V_{hmk}(r)=D_e\left(\frac{r-r_e}{r_e}\right)^2-\frac{V_0e^{-\alpha r}}{1-e^{-\alpha r}}\equiv\frac{B}{r}+\frac{C}{r^2}+D_e-\frac{V_0e^{-\alpha r}}{1-e^{-\alpha r}} \quad (6)$$

Where, D_e is the dissociation energy, r_e is the equilibrium inter-molecular separation, $V_0=Ze^2\alpha$ is the depth of the potential, α is the adjustable screening parameter, $B=-2r_eD_e$ and $C=D_er_e^2$. To achieve this goal of our current research it is useful to make a summary for the Klein–Gordon equation KGE with Hulthén–Kratzer potential model for a system of reduced mass μ of diatomic molecules such as N_2, I_2, CO, NO and HCl in 3-dimensional relativistic quantum mechanics [6]:

$$\left\{-\Delta+(\mu+S_{hmk}(r))^2-(E_{nl}-V_{hmk}(r))^2\right\}\Psi(r,\theta,\varphi)=0\Rightarrow$$

$$\left\{\begin{array}{l}\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\left(E_{nl}^2-\mu^2\right)-2\left(E_{nl}V_{hmk}(r)+\mu S_{hmk}(r)\right)\\ +V_{hmk}^2(r)-S_{hmk}^2(r)-\frac{l(l+1)}{r^2}\end{array}\right\}R_{nl}(r)=0 \quad (7)$$

The vector potential $V_{hmk}(r)$ is due to the four-vector linear momentum operator $A^{\mu}\left(V_{hmk}(r),\vec{A}=0\right)$ and the space–time scalar potential $S_{hmk}(r)$, E_{nl} represents the relativistic rotational-vibrational energy eigenvalues in 3-dimensions, n and l represents the vibrational and rotational quantum numbers, respectively. Since the Hulthén–Kratzer, potential model has spherical symmetry, allowing the solutions of the time-independent KGE of the known form $\Psi(r,\theta,\varphi)=R_{nl}(r)Y_l^m(\theta,\varphi)$, where $Y_l^m(\theta,\varphi)$ denotes the spherical harmonic function, and Δ is the ordinary 3-dimensional Laplacian operator. To eliminate the first order derivative, we introduce a new radial wave function to the form $U_{nl}(r)=rR_{nl}(r)$, thus Eqn. (7) become:

$$\left\{\begin{array}{l}\frac{d^2}{dr^2}-\left(\mu^2-E_{nl}^2\right)-2\left(E_{nl}V_{hmk}(r)+\mu S_{hmk}(r)\right)\\ +V_{hmk}^2(r)-S_{hmk}^2(r)-\frac{l(l+1)}{r^2}\end{array}\right\}U_{nl}(r)=0 \quad (8)$$

With the equal scalar and vector potential being taken as the generalized hyperbolic potential, $V_{hmk}(r)=S_{hmk}(r)$ we obtain the following second order Schrodinger-like equation:

$$\left\{\frac{d^2}{dr^2}-\left(E_{eff}^{hmk}+V_{eff}^{hmk}(r)\right)\right\}U_{nl}(r)=0 \quad (9)$$

The shorthand notation $V_{eff}^{hmk}(r)\equiv 2(E_{nl}+\mu)V_{hmk}(r)+\frac{l(l+1)}{r^2}$ and $E_{eff}^{hmk}\equiv\mu^2-E_{nl}^2$. The complete wave function as a function of the Jacobi polynomial and the spherical harmonic functions is given by [6]:

$$\Psi(r,\theta,\varphi)=B_n\frac{s^{\lambda_{nl}}}{r}(1-s)^{G_{nl}}P_n^{(2\lambda_{nl},2G_{nl}-1)}(1-2s)Y_l^m(\theta,\varphi) \quad (10)$$

Here $s=\exp(-\alpha r)$, $\lambda_{nl}=\sqrt{\varepsilon_{nl}^2+\Lambda_{nl}-\delta_{nl}}$, $G_{nl}=1/2+\sqrt{\frac{1}{4}+\Lambda_{nl}}$, $\Lambda_{nl}=\frac{D_er_e^2(E_{nl}+\mu)}{\alpha}+l(l+1)$, $\delta_{nl}=\frac{2\mu D_er_e(E_{nl}+\mu)}{\alpha}$, $-\varepsilon_{nl}^2=\frac{E_{nl}^2-\mu^2}{\alpha^2}-\frac{D_e}{\alpha^2}(E_{nl}+\mu)$ and B_n is the normalization constant. The relativistic energy E_{nl} of the potential in Eqn. (6) are given by [6]:

$$E_{nl}^2-\mu^2=l(l+1)\alpha^2+\left(D_er_e^2\alpha^2+D_e-2D_er_e\alpha\right)(E_{nl}+\mu)-\frac{\alpha^2}{4}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+l(l+1)+D_er_e^2(E_{nl}+\mu)}\right)^2+\left(D_er_e^2-\frac{V_0}{\alpha^2}-\frac{2D_er_e}{\alpha}(E_{nl}+\mu)+l(l+1)\right)}{n+\frac{1}{2}+\sqrt{\frac{1}{4}+l(l+1)+D_er_e^2(E_{nl}+\mu)}}\right]^2 \quad (11)$$

3. Solutions of MKGE under MHKP Model in (RNC:3D-RS) and (NRNC: 3D-RS) Symmetries

At the beginning of this section, we shall give and define a formula of modified Hulthén–Kratzer potential model in the symmetries of relativistic noncommutative three-dimensional real space (RNC: 3D-RS). To achieve this goal , it is useful to write the modified Klein-Gordon equation by applying the notion of Weyl-Moyal star product, which we have seen previously in the the Eqn. (3), on the differential equation that is satisfied by the radial wave function $U_l(r)$ in Eqn. (8), thus, the radial wave function $U_l(r)$ in (RNC: 3D-RS) symmetries become:

$$\left\{\frac{d^2}{dr^2}-\left(\mu^2-E_{nl}^2\right)-2\left(E_{nl}+\mu\right)V_{hmk}(r)-\frac{l(l+1)}{r^2}\right\}U_{nl}(r)=0\Rightarrow$$
$$\left\{\frac{d^2}{dr^2}-\left(\mu^2-E_{nl}^2\right)-2\left(E_{nl}+\mu\right)V_{hmk}(r)-\frac{l(l+1)}{r^2}\right\}*U_{nl}(r)=0 \tag{12}$$

We know that the Bopp's shift method has been applied effectively and has succeeded in simplifying three basic equations modified Schrödinger equation MSE ,MKGE and modified Dirac equation MDE with the notion of star product to the Schrödinger equation SE, KGE and Dirac equation DE with the notion of ordinary product, respectively [12-14,36,40-47]. The results of the application of this method were very useful and yielded promising results in many physical and chemical fields. The method reduced MSE, MKGE and MDE to the SE, KGE and DE, respectively, under the simultaneous translation in space. The CNCCRs with star product in Eqn. (2) become new CNCCRs without the notion of star product as follows [41-49]:

$$\left[\hat{x}_{\mu}^{S},\hat{x}_{\nu}^{S}\right]=\left[\hat{x}_{\mu}^{H}(t),\hat{x}_{\nu}^{H}(t)\right]=i\theta_{\mu\nu} \tag{13}$$

The generalized positions and momentum coordinates $\left(\hat{x}_{\mu}^{S},\hat{p}_{\mu}^{S}\right)$ and $\left(\hat{x}_{\mu}^{H},\hat{p}_{\mu}^{H}\right)(t)$ in the symmetries (RNC: 3D-RS) and (NRNC: 3D-RS) are defined in terms of the corresponding coordinates $\left(x_{\mu}^{S},p_{\mu}^{S}\right)$ and $\left(x_{\mu}^{H},p_{\mu}^{H}\right)$ via [43-51]:

$$\begin{cases}\left(x_{\mu}^{S},p_{\mu}^{S}\right)\Rightarrow\left(\hat{x}_{\mu}^{S},\hat{p}_{\mu}^{S}\right)=\left(x_{\mu}^{S}-\frac{\varepsilon_{\mu\nu}\theta}{2}p_{\nu}^{S},p_{\mu}^{S}\right)\\ \left(x_{\mu}^{H},p_{\mu}^{H}\right)\Rightarrow\left(\hat{x}_{\mu}^{H},\hat{p}_{\mu}^{H}\right)(t)=\left(x_{\mu}^{H}(t)-\frac{\varepsilon_{\mu\nu}\theta}{2}p_{\nu}^{H}(t),p_{\mu}^{H}(t)\right)\end{cases} \tag{14}$$

This allows us to find the operator $r_d^2\Rightarrow\left(r_{nc}^d\right)^2=r_d^2-\vec{\mathbf{L}}\vec{\Theta}$ in the symmetries of (RNC: 3D-RS) and (NRNC: 3D-RS) [43-47], with r_{nc}^d denote to the diatomic molecule distance in NCQM. It is convenient to introduce a shorthand notation which will save us a lot of writing $r_{nc}^d\to\hat{r}$ and $r_d^2\to r^2$. In this notation the previously relation reduced to $r^2\Rightarrow\hat{r}^2=r^2-\vec{\mathbf{L}}\vec{\Theta}$. The coupling $\vec{\mathbf{L}}\vec{\Theta}$ equal $\left(L_x\Theta_{12}+L_y\Theta_{23}+L_z\Theta_{13}\right)$, here L_x, L_y and L_z are present the usually components of angular momentum operator $\vec{\mathbf{L}}$ while the new non-commutative parameter $\Theta_{\mu\nu}$ equal $\theta_{\mu\nu}/2$. According to the Bopp shift method, Eqn. (12) becomes similarly to the following like the Schrödinger equation [12,13,23] (without the notions of star product):

$$\left\{\frac{d^2}{dr^2}-\left(\mu^2-E_{nl}^2\right)-2\left(E_{nl}+\mu\right)V_{hmk}(r)-\frac{l(l+1)}{r^2}\right\}*U_{nl}(r)=0\Rightarrow$$
$$\left\{\frac{d^2}{dr^2}-\left(\mu^2-E_{nl}^2\right)-2\left(E_{nl}+\mu\right)V_{hmk}(\hat{r})-\frac{l(l+1)}{\hat{r}^2}\right\}U_{nl}(r)=0 \tag{15}$$

The new operator $V_{hmk}(\hat{r})$ can be expressed as:

$$V_{hmk}(\hat{r})=V_{hmk}(r)-\frac{\vec{\mathbf{L}}\vec{\Theta}}{2r}\frac{\partial V_{hmk}(r)}{\partial r}+O\left(\Theta^2\right) \tag{16}$$

After straightforward calculations, we can obtain the following results:

$$\frac{\partial V_{hmk}(r)}{\partial r}=-\frac{B}{r^2}-\frac{2C}{r^3}+\frac{V_0\alpha e^{-\alpha r}}{1-e^{-\alpha r}}+\frac{V_0\alpha e^{-2\alpha r}}{\left(1-e^{-\alpha r}\right)^2}\quad\text{And}$$
$$\frac{1}{\hat{r}^2}\approx\frac{1}{r^2}+\frac{\vec{\mathbf{L}}\vec{\Theta}}{r^4}+O\left(\Theta^2\right) \tag{17}$$

Which, allows us to obtain

$$V_{hmk}(\hat{r})=V_{hmk}(r)+V_{hmk}^{pert}(r)+O\left(\Theta^2\right) \tag{18}$$

Where

$$V_{hmk}^{pert}(r)=\left(\frac{B}{2r^3}+\frac{C}{r^4}-\frac{2V_0\alpha}{r}\frac{e^{-\alpha r}}{1-e^{-\alpha r}}-\frac{2V_0\alpha}{r}\frac{e^{-2\alpha r}}{\left(1-e^{-\alpha r}\right)^2}\right)\vec{\mathbf{L}}\vec{\Theta} \tag{19}$$

So we can rewrite the new modified radial part (new differential equation) of the MKGE in the symmetries of (RNC: 3D-RS) as follows:

$$\left\{\begin{array}{l}\frac{d^2}{dr^2}-\left(\mu^2-E_{nl}^2\right)-2\left(E_{nl}+\mu\right)V_{hmk}(r)-\frac{l(l+1)}{r^2}\\ -2\left(E_{nl}+\mu\right)V_{hmk}^{pert}(r)-\frac{l(l+1)}{r^4}\vec{\mathbf{L}}\vec{\Theta}\end{array}\right\}U_{nl}(r)=0 \tag{20}$$

Moreover, to illustrate the above equation in a simple mathematical way and attractive form, it is useful to enter the following symbol $V_{nc-eff}^{hmk}(r)$, thus the radial Eqn. (20) becomes:

$$\left\{\frac{d^2}{dr^2}-\left[E_{eff}^{hmk}+V_{nc-eff}^{hmk}(r)\right]\right\}U_{nl}(r)=0, \tag{21}$$

With:

$$V_{nc-eff}^{hmk}(r)=V_{eff}^{hmk}(r)+V_{pert-eff}^{hmk}(r) \tag{22}$$

Where, $V_{pert-eff}^{hmk}(r)$ is given by the following relation:

$$V_{pert-eff}^{hmk}(r)=\frac{l(l+1)}{r^4}\vec{\mathbf{L}}\vec{\Theta}+2(E_{nl}+\mu)V_{hmk}^{pert}(r) \quad (23)$$

By making the substitution Eqn. (19) into Eqn. (21), we find $V_{pert-eff}^{hmk}(r)$ in the symmetries of (RNC: 3D-RSP) as follows:

$$V_{pert-eff}^{hmk}(r)=\left[\begin{array}{c}\frac{l(l+1)+2(E_{nl}+\mu)C}{r^4}+2(E_{nl}+\mu)\\ \left(\frac{B}{2r^3}-\frac{2V_0\alpha}{r}\frac{e^{-\alpha r}}{1-e^{-\alpha r}}-\frac{2V_0\alpha}{r}\frac{e^{-2\alpha r}}{\left(1-e^{-\alpha r}\right)^2}\right)\end{array}\right]\vec{\mathbf{L}}\vec{\Theta} \quad (24)$$

The Eqn. (21) cannot be solved analytically for any state because of the centrifugal term and the studied potential itself. Therefore, in the present work, we considered the following approximation type suggested by Greene and Aldrich and Dong *et al.* for them [6, 53-55]:

$$\frac{1}{r^2}\approx\frac{\alpha^2\exp(-\alpha r)}{(1-\exp(-\alpha))^2} \text{ and } \frac{1}{r}\approx\frac{\alpha\exp(-\alpha/2r)}{1-\exp(-\alpha/2r)} \quad (25.1)$$

This allows us to obtain the following results:

$$\begin{cases}\frac{1}{r}\approx\frac{\alpha\exp(-\alpha/2r)}{1-\exp(-\alpha r)}=\frac{\alpha s^{1/2}}{1-s}\\ \frac{1}{r^3}\approx\frac{\alpha^3\exp(-3\alpha/2r)}{(1-\exp(-\alpha r))^3}=\frac{\alpha^3 s^{3/2}}{(1-s)^3}\\ \frac{1}{r^4}\approx\frac{\alpha^4\exp(-2\alpha r)}{(1-\exp(-\alpha r))^4}=\frac{\alpha^4 s^2}{(1-s)^4}\end{cases} \quad (25.2)$$

This gives the perturbative effective potential as follows:

$$V_{pert-eff}^{hmk}(r)=\left[\frac{X_{nl}s^2}{(1-s)^4}+2(E_{nl}+\mu)\left(\frac{B\alpha^3 s^{3/2}}{2(1-s)^3}-\frac{2V_0\alpha^2 s^{3/2}}{(1-s)^2}-\frac{2V_0\alpha^2 s^{5/2}}{(1-s)^3}\right)\right]\vec{\mathbf{L}}\vec{\Theta} \quad (26)$$

Where, $X_{nl}=\alpha^4\left(l(l+1)+2(E_{nl}+\mu)C\right)$.This allows to applying standard perturbation theory to determine the nonrelativistic energy shift ΔE_{hmk} of diatomic molecules such as N_2, I_2, CO, NO and HCl at first order of the infinitesimal parameter Θ due to non-commutativity of space-space properties. The Hulthén–Kratzer potential model is extended by including new terms proportional with the radial terms ($1/r$, $1/r^3$ and $1/r^4$) to becomes MHKP model in (RNC-3D: RSP) and (NRNC-3D: RSP) symmetries. The additive part $V_{pert-eff}^{hmk}(r)$ of the new effective potential $V_{nc-eff}^{hmk}(r)$ is proportional to the infinitesimal vector $\vec{\Theta}=\Theta_{11}e_x+\Theta_{12}e_y+\Theta_{13}e_z$. This allows us to consider physically that the additive effective potential $V_{pert-eff}^{hmk}(r)$ as a perturbation potential compared to the main potential (parent potential operator $V_{eff}^{hmk}(r)$) in the symmetries of (RNC: 3D-RS) and (NRNC-3D: RSP), that is, the inequality $V_{pert-eff}^{hmk}(r)\prec\prec V_{eff}^{hmk}(r)$ has become achieved. That is, all the physical justifications for applying the time-independent perturbation theory become satisfied. This allows us to give a complete prescription for determing the energy level of the generalized n^{th} excited states. Now, we apply the perturbative theory, in the case of RNCQM, we find the expectation values $\frac{s^2}{(1-s)^4}$, $\frac{s^{3/2}}{(1-s)^3}$, $\frac{s^{3/2}}{(1-s)^2}$ and $\frac{s^{5/2}}{(1-s)^3}$ taking into account the wave function which we have seen previously in the Eqn. (10). After straightforward calculations we obtain the following results:

$$\begin{aligned}
\left\langle n,l,m\left|\frac{s^2}{(1-s)^4}\right|n,l,m\right\rangle&=B_n^2\int_0^{+\infty}s^{2\lambda_{nl}+2}(1-s)^{2G_{nl}-4}\left[P_n^{(2\lambda_{nl},2G_{nl}-1)}(1-2s)\right]^2dr\\
\left\langle n,l,m\left|\frac{s^{3/2}}{(1-s)^3}\right|n,l,m\right\rangle&=B_n^2\int_0^{+\infty}s^{2\lambda_{nl}+3/2}(1-s)^{2G_{nl}-3}\left[P_n^{(2\lambda_{nl},2G_{nl}-1)}(1-2s)\right]^2dr\\
\left\langle n,l,m\left|\frac{s^{3/2}}{(1-s)^2}\right|n,l,m\right\rangle&=B_n^2\int_0^{+\infty}s^{2\lambda_{nl}+3/2}(1-s)^{2G_{nl}-2}\left[P_n^{(2\lambda_{nl},2G_{nl}-1)}(1-2s)\right]^2dr\\
\left\langle n,l,m\left|\frac{s^{5/2}}{(1-s)^3}\right|n,l,m\right\rangle&=B_n^2\int_0^{+\infty}s^{2\lambda_{nl}+5/2}(1-s)^{2G_{nl}-3}\left[P_n^{(2\lambda_{nl},2G_{nl}-1)}(1-2s)\right]^2dr
\end{aligned} \quad (27)$$

Where $s=\exp(-\alpha r)$, this allows us to obtain $dr=-\frac{1}{\alpha}\frac{ds}{s}$. After introducing a new variable $z=1-2s$, we have $dr=\frac{1}{\alpha}\frac{dz}{1-z}$, $s=\frac{1-z}{2}$ and $1-s=\frac{z+1}{2}$, the approximations Eqn. (27) in that case have the following form:

$$\begin{aligned}
\left\langle n,l,m\left|\frac{s^2}{(1-s)^4}\right|n,l,m\right\rangle&=\frac{B_n^2}{2^{2\lambda_{nl}+2G_{nl}-2}\alpha}\int_{-1}^{+1}(1-z)^{2\lambda_{nl}+1}(1+z)^{2G_{nl}-4}\left[P_n^{(2\lambda_{nl},2G_{nl}-1)}(z)\right]^2dz\\
\left\langle n,l,m\left|\frac{s^{3/2}}{(1-s)^3}\right|n,l,m\right\rangle&=\frac{B_n^2}{2^{2\lambda_{nl}+2G_{nl}-3/2}\alpha}\int_{-1}^{+1}(1-z)^{2\lambda_{nl}+1/2}(1+z)^{2G_{nl}-3}\left[P_n^{(2\lambda_{nl},2G_{nl}-1)}(z)\right]^2dz\\
\left\langle n,l,m\left|\frac{s^{3/2}}{(1-s)^2}\right|n,l,m\right\rangle&=\frac{B_n^2}{2^{2\lambda_{nl}+2G_{nl}-1/2}\alpha}\int_{-1}^{+1}(1-z)^{2\lambda_{nl}+1/2}(1+z)^{2G_{nl}-2}\left[P_n^{(2\lambda_{nl},2G_{nl}-1)}(z)\right]^2dz\\
\left\langle n,l,m\left|\frac{s^{5/2}}{(1-s)^3}\right|n,l,m\right\rangle&=\frac{B_n^2}{2^{2\lambda_{nl}+2G_{nl}-1/2}\alpha}\int_{-1}^{+1}(1-z)^{2\lambda_{nl}+3/2}(1+z)^{2G_{nl}-3}\left[P_n^{(2\lambda_{nl},2G_{nl}-1)}(z)\right]^2dz
\end{aligned} \quad (28)$$

We have applied the property of the spherical harmonics, which has the form $\int Y_l^m(\theta,\varphi)Y_{l'}^{m'}(\theta,\varphi)\sin(\theta)d\theta d\varphi=\delta_{ll'}\delta_{mm'}$. For relieving the burden of writing, we will provide useful abbreviation $\langle n,l,m|A|n,l,m\rangle\equiv\langle A\rangle_{(n,l,m)}$. For the ground state $n=0$, we have $P_{n=0}^{(2\lambda_{0l},2G_{0l}-1)}(z)=1$, thus the expectation values in Eqn. (28) reduce to the following simple form:

$$\left\langle \frac{s^2}{(1-s)^4} \right\rangle_{(0,l,m)} = \frac{B_0^2}{2^{2\lambda_{0l}+2G_{0l}-2}\alpha}\int_{-1}^{+1}(1-z)^{2\lambda_{0l}+1}(1+z)^{2G_{0l}-4}dz$$
$$\left\langle \frac{s^{3/2}}{(1-s)^3} \right\rangle_{(0,l,m)} = \frac{B_0^2}{2^{2\lambda_{0l}+2G_{0l}-3/2}\alpha}\int_{-1}^{+1}(1-z)^{2\lambda_{0l}+1/2}(1+z)^{2G_{0l}-3}dz \quad (29)$$
$$\left\langle \frac{s^{3/2}}{(1-s)^2} \right\rangle_{(0,l,m)} = \frac{B_0^2}{2^{2\lambda_{0l}+2G_{0l}-1/2}\alpha}\int_{-1}^{+1}(1-z)^{2\lambda_{0l}+1/2}(1+z)^{2G_{0l}-2}dz$$
$$\left\langle \frac{s^{5/2}}{(1-s)^3} \right\rangle_{(0,l,m)} = \frac{B_0^2}{2^{2\lambda_{0l}+2G_{0l}-1/2}\alpha}\int_{-1}^{+1}(1-z)^{2\lambda_{0l}+3/2}(1+z)^{2G_{0l}-3}dz$$

Where $\lambda_{0l}=\sqrt{\varepsilon_{0l}^2+\Lambda_{0l}-\delta_{0l}}$, $G_{0l}=1/2+\sqrt{\frac{1}{4}+\Lambda_{0l}}$, $\Lambda_{0l}=\frac{D_e r_e^2(E_{0l}+\mu)}{\alpha}+l(l+1)$, $\delta_{0l}=\frac{2\mu D_e r_e(E_{0l}+\mu)}{\alpha}$, $-\varepsilon_{0l}^2=\frac{E_{0l}^2-\mu^2}{\alpha^2}-\frac{D_e}{\alpha^2}(E_{0l}+\mu)$ and E_{0l} obtained from :

$$E_{0l}^2-\mu^2=l(l+1)\alpha^2+\left(D_e r_e^2\alpha^2+D_e-2D_e r_e\alpha\right)(E_{0l}+\mu)-\frac{\alpha^2}{4}\left[\frac{\left(\frac{1}{2}+\sqrt{\frac{1}{4}+l(l+1)+D_e r_e^2(E_{0l}+\mu)}\right)^2+\left(D_e r_e^2-\frac{V_0}{\alpha^2}-\frac{2D_e r_e}{\alpha}(E_{0l}+\mu)+l(l+1)\right)}{\frac{1}{2}+\sqrt{\frac{1}{4}+l(l+1)+D_e r_e^2(E_{0l}+\mu)}}\right]^2 \quad (30)$$

Comparing Eqn. (29) with the integral of the form [56]:

$$\int_{-1}^{+1}(1-p)^{\alpha}(1+p)^{\beta}P_m^{(\alpha,\beta)}(p)P_n^{(\alpha,\beta)}(p)dp=\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}\delta_{mn}\Rightarrow\int_{-1}^{+1}(1-p)^{n+\alpha}(1+p)^{n+\beta}dp=\frac{2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+1)}\text{ for }(n=0,1,) \quad (31)$$

We obtain the expectation values as:

$$\left\langle \frac{s^2}{(1-s)^4} \right\rangle_{(0,l,m)} = \frac{B_0^2}{\alpha}\frac{\Gamma(2\lambda_{0l}+2)\Gamma(2G_{0l}-3)}{(2\lambda_{0l}+2G_{0l}-2)\Gamma(2\lambda_{0l}+2G_{0l}-2)}$$
$$\left\langle \frac{s^{3/2}}{(1-s)^3} \right\rangle_{(0,l,m)} = \frac{B_0^2}{\alpha}\frac{\Gamma(2\lambda_{0l}+3/2)\Gamma(2G_{0l}-2)}{(2\lambda_{0l}+2G_{0l}-3/2)\Gamma(2\lambda_{0l}+2G_{0l}-3/2)} \quad (32)$$
$$\left\langle \frac{s^{3/2}}{(1-s)^2} \right\rangle_{(0,l,m)} = \frac{B_0^2}{\alpha}\frac{\Gamma(2\lambda_{0l}+3/2)\Gamma(2G_{0l}-1)}{(2\lambda_{0l}+2G_{0l}+1/2)\Gamma(2\lambda_{0l}+2G_{0l}+1/2)}$$
$$\left\langle \frac{s^{5/2}}{(1-s)^3} \right\rangle_{(0,l,m)} = \frac{B_0^2}{\alpha}\frac{\Gamma(2\lambda_{0l}+5/2)\Gamma(2G_{0l}-2)}{(2\lambda_{0l}+2G_{0l}-1/2)\Gamma(2\lambda_{0l}+2G_{0l}-1/2)}$$

For the first excited state $n=1$, we have $P_1^{(2\lambda_{1l},2G_{1l}-1)}(z)=a-b(1-z)$, the expectation values in Eqn. (28) reduce to the following simple form:

$$\left\langle \frac{s^2}{(1-s)^4} \right\rangle_{(1,l,m)} = \frac{B_1^2}{2^{2\lambda_{1l}+2G_{1l}-2}\alpha}\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+1}(1+z)^{2G_{1l}-4}[a-b(1-z)]^2dz$$
$$\left\langle \frac{s^{3/2}}{(1-s)^3} \right\rangle_{(1,l,m)} = \frac{B_n^2}{2^{2\lambda_{1l}+2G_{1l}-3/2}\alpha}\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+1/2}(1+z)^{2G_{1l}-3}[a-b(1-z)]^2dz \quad (33)$$
$$\left\langle \frac{s^{3/2}}{(1-s)^2} \right\rangle_{(1,l,m)} = \frac{B_n^2}{2^{2\lambda_{1l}+2G_{1l}-1/2}\alpha}\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+1/2}(1+z)^{2G_{1l}-2}[a-b(1-z)]^2dz$$
$$\left\langle \frac{s^{5/2}}{(1-s)^3} \right\rangle_{(1,l,m)} = \frac{B_n^2}{2^{2\lambda_{1l}+2G_{1l}-1/2}\alpha}\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+3/2}(1+z)^{2G_{1l}-3}[a-b(1-z)]^2dz$$

With $a=2\lambda_{1l}+1$, $b=(\lambda_{1l}+G_{1l}+\frac{1}{2})$, $\lambda_{1l}=\sqrt{\varepsilon_{1l}^2+\Lambda_{1l}-\delta_{1l}}$, $G_{1l}=1/2+\sqrt{\frac{1}{4}+\Lambda_{1l}}$, $\Lambda_{1l}=\frac{D_e r_e^2(E_{1l}+\mu)}{\alpha}+l(l+1)$, $\delta_{1l}=\frac{2\mu D_e r_e(E_{1l}+\mu)}{\alpha}$, $-\varepsilon_{1l}^2=\frac{E_{1l}^2-\mu^2}{\alpha^2}-\frac{D_e}{\alpha^2}(E_{1l}+\mu)$ and E_{1l} obtained from :

$$E_{1l}^2-\mu^2=l(l+1)\alpha^2+\left(D_e r_e^2\alpha^2+D_e-2D_e r_e\alpha\right)(E_{1l}+\mu)-\frac{\alpha^2}{4}\left[\frac{\left(\frac{3}{2}+\sqrt{\frac{1}{4}+l(l+1)+D_e r_e^2(E_{0l}+\mu)}\right)^2+\left(D_e r_e^2-\frac{V_0}{\alpha^2}-\frac{2D_e r_e}{\alpha}(E_{1l}+\mu)+l(l+1)\right)}{\frac{3}{2}+\sqrt{\frac{1}{4}+l(l+1)+D_e r_e^2(E_{1l}+\mu)}}\right]^2 \quad (34)$$

A direct simplification to Eqn. (33) gives:

$$\left\langle \frac{s^2}{(1-s)^4} \right\rangle_{(1,l,m)} = \frac{B_1^2}{2^{2\lambda_{1l}+2G_{1l}-2}\alpha}\left(a^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+1}(1+z)^{2G_{1l}-4}dz - 2ab\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+2}(1+z)^{2G_{1l}-4}dz + b^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+3}(1+z)^{2G_{1l}-4}dz\right)$$
$$\left\langle \frac{s^{3/2}}{(1-s)^3} \right\rangle_{(1,l,m)} = \frac{B_n^2}{2^{2\lambda_{1l}+2G_{1l}-3/2}\alpha}\left(a^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+1/2}(1+z)^{2G_{1l}-3}dz - 2ab\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+3/2}(1+z)^{2G_{1l}-3}dz + b^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+5/2}(1+z)^{2G_{1l}-3}dz\right) \quad (35.1)$$

$$\left\langle \frac{s^{3/2}}{(1-s)^2} \right\rangle_{(1,l,m)} = \frac{B_n^2}{2^{2\lambda_{1l}+2G_{1l}-1/2}\alpha}\left(a^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+1/2}(1+z)^{2G_{1l}-2}dz - 2ab\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+3/2}(1+z)^{2G_{1l}-2}dz + b^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+5/2}(1+z)^{2G_{1l}-2}dz\right)$$
$$\left\langle \frac{s^{5/2}}{(1-s)^3} \right\rangle_{(1,l,m)} = \frac{B_n^2}{2^{2\lambda_{1l}+2G_{1l}-1/2}\alpha}\left(a^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+3/2}(1+z)^{2G_{1l}-3}dz - 2ab\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+5/2}(1+z)^{2G_{1l}-3}dz + b^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+5/2}(1+z)^{2G_{1l}-3}dz\right) \quad (35.2)$$

Comparing Eqns. (35.1) and (35.2) with the integral (Eqn. (31)), we have the expectation values as:

$$\left\langle \frac{s^2}{(1-s)^4} \right\rangle_{(1,l,m)} = \frac{B_1^2}{\alpha}\left(\frac{a^2\Gamma(2\lambda_{1l}+2)\Gamma(2G_{1l}-3)}{(D_{1l}-2)\Gamma(D_{1l}-2)} - \frac{4ab\Gamma(2\lambda_{1l}+3)\Gamma(2G_{1l}-3)}{(D_{1l}-1)\Gamma(D_{1l}-1)} + \frac{4b^2\Gamma(2\lambda_{1l}+4)\Gamma(2G_{1l}-3)}{D_{1l}\Gamma(D_{1l})}\right)$$
$$\left\langle \frac{s^{3/2}}{(1-s)^3} \right\rangle_{(1,l,m)} = \frac{B_1^2}{\alpha}\left(\frac{a^2\Gamma(2\lambda_{1l}+3/2)\Gamma(2G_{1l}-2)}{(D_{1l}-3/2)\Gamma(D_{1l}-3/2)} - \frac{4ab\Gamma(2\lambda_{1l}+5/2)\Gamma(2G_{1l}-2)}{(D_{1l}-1/2)\Gamma(D_{1l}-1/2)} + \frac{4b^2\Gamma(2\lambda_{1l}+7/2)\Gamma(2G_{1l}-2)}{(D_{1l}+1/2)\Gamma(D_{1l}+1/2)}\right) \quad (36.1)$$

$$\left\langle \frac{s^{3/2}}{(1-s)^2} \right\rangle_{(1,l,m)} = \frac{B_1^2}{\alpha}\left(\frac{a^2\Gamma(2\lambda_{1l}+3/2)\Gamma(2G_{1l}-1)}{(D_{1l}-1/2)\Gamma(D_{1l}-1/2)} - \frac{4ab\Gamma(2\lambda_{1l}+5/2)\Gamma(2G_{1l}-1)}{(D_{1l}+1/2)\Gamma(D_{1l}+1/2)} + \frac{4b^2\Gamma(2\lambda_{1l}+7/2)\Gamma(2G_{1l}-1)}{(D_{1l}+3/2)\Gamma(D_{1l}+3/2)}\right)$$
$$\left\langle \frac{s^{5/2}}{(1-s)^3} \right\rangle_{(1,l,m)} = \frac{B_1^2}{\alpha}\left(\frac{a^2\Gamma(2\lambda_{1l}+5/2)\Gamma(2G_{1l}-2)}{(D_{1l}-1/2)\Gamma(D_{1l}-1/2)} - \frac{4ab\Gamma(2\lambda_{1l}+7/2)\Gamma(2G_{1l}-2)}{(D_{1l}+1/2)\Gamma(D_{1l}+1/2)} + \frac{4b^2\Gamma(2\lambda_{1l}+7/2)\Gamma(2G_{1l}+1-2)}{(D_{1l}+1/2)\Gamma(D_{1l}+1/2)}\right) \quad (36.2)$$

Where, $D_{1l} = 2\lambda_{1l} + 2G_{1l}$. Our current research is divided into two main parts, the first part corresponds to replace the coupling of angular momentum operator with non-commutativity properties $\vec{\mathbf{L}}\vec{\Theta}$ by the new equivalent coupling $\Theta\vec{L}\vec{S}$ (with $\Theta = \left(\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2\right)^{1/2}$), we have chosen the vector $\vec{\Theta}$ parallel to the spin $\vec{S}$ of diatomic molecules such as (N_2, I_2, CO, NO and HCl) and then we replace $\Theta\vec{L}\vec{S}$ by $\frac{\Theta}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$. Furthermore, in the quantum mechanics the operators ($\hat{H}_{nc-r}^{hmk}$, J^2, L^2, S^2 and J_z) forms a complete set of conserved physics quantities CCPQ, the eigenvalues of the operator $\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$ are equal the values $j(j+1) - l(l+1) - s(s+1)$, with $|l-s| \le j \le |l+s|$. Consequently, the energy shift $E(n=0,j,l,s)$ and $E(n=1,j,l,s)$ due to the perturbed spin-orbit coupling which produced by the effect of perturbed effective potential $V_{pert-eff}^{hmk}(r)$ for the ground state, the first excited state in (RNC: 3D-RS) symmetries as follows:

$$E(n=0,j,l,s) = k(l)\left\{\left\langle \frac{s^2}{(1-s)^4} \right\rangle_{(0,l,m)} X_{\grave{a}l} + 2(E_{nl}+\mu)\left(\frac{B\alpha^3}{2}\left\langle \frac{s^{3/2}}{(1-s)^3} \right\rangle_{(0,l,m)} - 2V_0\alpha^2\left\langle \frac{s^{3/2}}{(1-s)^2} \right\rangle_{(0,l,m)} - 2V_0\alpha^2\left\langle \frac{s^{5/2}}{(1-s)^3} \right\rangle_{(0,l,m)}\right)\right\}$$
$$\Delta E(n=1,j,l,s) = k(l)\left\{\left\langle \frac{s^2}{(1-s)^4} \right\rangle_{(1,l,m)} X_{\grave{a}l} + 2(E_{nl}+\mu)\left(\frac{B\alpha^3}{2}\left\langle \frac{s^{3/2}}{(1-s)^3} \right\rangle_{(1,l,m)} - 2V_0\alpha^2\left\langle \frac{s^{3/2}}{(1-s)^2} \right\rangle_{(1,l,m)} - 2V_0\alpha^2\left\langle \frac{s^{5/2}}{(1-s)^3} \right\rangle_{(1,l,m)}\right)\right\} \quad (37)$$

Where, $k(l) \equiv \frac{1}{2}\{j(j+1) - l(l+1) - s(s+1)\}$. Which can be generalized easily to the n^{th} excited states in (RNC: 3D-RS) symmetries as follows:

$$\Delta E(n,j,l,s) = k(l)\left\{\left\langle \frac{s^2}{(1-s)^4} \right\rangle_{(n,l,m)} X_{\grave{a}l} + 2(E_{nl}+\mu)\left(\frac{B\alpha^3}{2}\left\langle \frac{s^{3/2}}{(1-s)^3} \right\rangle_{(n,l,m)} - 2V_0\alpha^2\left\langle \frac{s^{3/2}}{(1-s)^2} \right\rangle_{(n,l,m)} - 2V_0\alpha^2\left\langle \frac{s^{5/2}}{(1-s)^3} \right\rangle_{(n,l,m)}\right)\right\} \quad (38)$$

The second is corresponding to replace both ($\vec{\mathbf{L}}\vec{\Theta}$ and Θ_{12}) by ($\sigma_{12}\aleph L_z$ and $\sigma_{12}\aleph$, respectively), we have also need to apply $\langle n,l,m|L_z|n',l',m'\rangle = m'\delta_{nn'}\delta_{ll'}\delta_{mm'}$ (with $-(l,l')\leq(m,m')\leq+(l,l')$). All of this data allow for the discovery the new energy shift $\Delta E_{hmk}(n,m)$ due to the modified perturbed Zeeman effect which generated by influence of the perturbed effective potential for the ground state, the first excited state in (RNC: 3D-RS) symmetries as follows:

$$\Delta E(n=0,m)=\aleph\left\{\left\langle\frac{s^2}{(1-s)^4}\right\rangle_{(0,l,m)}X_{0l}+2(E_{nl}+\mu)\left(\frac{B\alpha^3}{2}\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(0,l,m)}-2V_0\alpha^2\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(0,l,m)}-2V_0\alpha^2\left\langle\frac{s^{5/2}}{(1-s)^3}\right\rangle_{(0,l,m)}\right)\right\}\sigma m \tag{39}$$

$$\Delta E(n=1,m)=\aleph\left\{\left\langle\frac{s^2}{(1-s)^4}\right\rangle_{(1,l,m)}X_{0l}+2(E_{nl}+\mu)\left(\frac{B\alpha^3}{2}\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(1,l,m)}-2V_0\alpha^2\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(1,l,m)}-2V_0\alpha^2\left\langle\frac{s^{5/2}}{(1-s)^3}\right\rangle_{(1,l,m)}\right)\right\}\sigma m$$

Which can be generalized easily to the n^{th} excited states in (RNC: 3D-RS) symmetries as follows:

$$\Delta E(n,m)=\aleph\left\{\left\langle\frac{s^2}{(1-s)^4}\right\rangle_{(n,l,m)}X_{0l}+2(E_{nl}+\mu)\left(\frac{B\alpha^3}{2}\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(n,l,m)}-2V_0\alpha^2\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(n,l,m)}-2V_0\alpha^2\left\langle\frac{s^{5/2}}{(1-s)^3}\right\rangle_{(n,l,m)}\right)\right\}\sigma m \tag{40}$$

4. **Results and Discussions**

In this part, we report our results on based to the superposition principle, which permitted to deduce the additive energy shift $\Delta E(n=0,j,l,s,m)$ and $\Delta E(n=1,j,l,s,m)$ due to the spin-orbital complying and modified Zeeman effect, which is induced by $V_{eff}^{hmk}(r)$ for the ground state, the first excited state in (RNC: 3D-RS) symmetries as follows:

$$\Delta E(n=0,j,l,s,m)=\left\{\left\langle\frac{s^2}{(1-s)^4}\right\rangle_{(0,l,m)}X_{\grave{a}l}+2(E_{nl}+\mu)\left(\frac{B\alpha^3}{2}\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(0,l,m)}-2V_0\alpha^2\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(0,l,m)}-2V_0\alpha^2\left\langle\frac{s^{5/2}}{(1-s)^3}\right\rangle_{(0,l,m)}\right)\right\}\{k(l)\Theta+\aleph\sigma m\} \tag{41}$$

$$\Delta E(n=1,j,l,s,m)=\left\{\left\langle\frac{s^2}{(1-s)^4}\right\rangle_{(1,l,m)}X_{\grave{a}l}+2(E_{nl}+\mu)\left(\frac{B\alpha^3}{2}\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(1,l,m)}-2V_0\alpha^2\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(1,l,m)}-2V_0\alpha^2\left\langle\frac{s^{5/2}}{(1-s)^3}\right\rangle_{(1,l,m)}\right)\right\}\{k(l)\Theta+\aleph\sigma m\}$$

This can be generalized easily to the n^{th} excited states in (RNC: 3D-RS) symmetries as follows:

$$\Delta E_{hmk}(n,j,l,s,m)=\left\{\aleph\left\{\left\langle\frac{s^2}{(1-s)^4}\right\rangle_{(n,l,m)}X_{\grave{a}l}+2(E_{nl}+\mu)\left(\frac{B\alpha^3}{2}\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(n,l,m)}-2V_0\alpha^2\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(n,l,m)}-2V_0\alpha^2\left\langle\frac{s^{5/2}}{(1-s)^3}\right\rangle_{(n,l,m)}\right)\right\}\sigma m\right\}\{k(l)\Theta+\aleph\sigma m\} \tag{42}$$

The above results present the energy shift, which is generated by the effect of non-commutative properties of space-space; it depended explicitly with the non-commutative parameters (Θ,σ). It is should be noted that the obtained effective energy $\Delta E_{hmk}(n,j,l,s,m)$ under the modified Hulthén–Kratzer potential model have a carry unit of energy because it resulted from the perturbed effective energy ($\mu^2-E_{nl}^2$) combined with the same energy value square and mass square, where we have the principle of equivalence between mass and energy at higher energy. This allows us to conclude the energy $E_{r-nc}^{hmk}(V_0,\alpha,r_e,D_e,n,j,l,s,m)$, in the symmetries of (RNC: 3D-RS), corresponding the generalized n^{th} excited states, as a functions of the shift energy $\Delta E_{hmk}(n,j,l,s,m)$ and E_{nl} due to the effect of Hulthén–Kratzer potential model in RQM, as follows:

$$E_{r-nc}^{hmk}(V_0,\alpha,r_e,D_e,n,j,l,s,m)=\mu+E_{nl}+\left[\left\{\left\langle\frac{s^2}{(1-s)^4}\right\rangle_{(n,l,m)}X_{nl}+2(E_{nl}+\mu)\left(\frac{B\alpha^3}{2}\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(n,l,m)}-2V_0\alpha^2\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(n,l,m)}-2V_0\alpha^2\left\langle\frac{s^{5/2}}{(1-s)^3}\right\rangle_{(n,l,m)}\right)\right\}\{k(l)\Theta+\aleph\sigma m\}\right]^{1/2} \tag{43}$$

Where E_{nl} is the relativistic energy in RQM, which obtained from Eqn. (11). Now, we want to apply Eq. (30) on the diatomic molecules N_2, I_2, CO, NO and HCl with non-null spin, for the special case $\vec{S}=\vec{1}$, we have $|l-1|\leq j\leq|l+1|$, thus we have three values of $j=l\pm1,l$, allows us the corresponding three values $(k_1(l),k_2(l),k_3(l))\equiv\frac{1}{2}(l,-2,-2l-2)$ and thus, we three values of energy:

$$\begin{aligned}&E^{hmk}_{r-nc}\left(k_1(l),V_0,\alpha,r_e,D_e,n,j,l=l+1,s,l,m\right)=\mu+E_{nl}+\left[\Xi\left(E_{nl},n,l,V_0,\alpha,r_e,D_e\right)\left\{\frac{l}{2}\Theta+\aleph\sigma m\right\}\right]^{1/2}\\&E^{hmk}_{r-nc}\left(k_2(l),V_0,\alpha,r_e,D_e,n,j,l=l,s,l,m\right)=\mu+E_{nl}+\left[\Xi\left(E_{nl},n,l,V_0,\alpha,r_e,D_e\right)\left\{-\Theta+\aleph\sigma m\right\}\right]^{1/2}\\&E^{hmk}_{r-nc}\left(k_3(l),V_0,\alpha,r_e,D_e,n,j,l=l-1,s,l,m\right)=\mu+E_{nl}+\left[\Xi\left(E_{nl},n,l,V_0,\alpha,r_e,D_e\right)\left\{-\frac{l+1}{2}\Theta+\aleph\sigma m\right\}\right]^{1/2}\end{aligned}\quad(44)$$

The new factor $\Xi\left(E_{nl},n,l,V_0,\alpha,r_e,D_e\right)$ is determined from the following expression:

$$\Xi(E_{nl},n,l,V_0,\alpha,r_e,D_e)\equiv\left\{\left\langle\frac{s^2}{(1-s)^4}\right\rangle_{(n,l,m)}X_{\grave{a}l}+2(E_{nl}+\mu)\left(\frac{B\alpha^3}{2}\left\langle\frac{s^{3/2}}{(1-s)^3}\right\rangle_{(n,l,m)}-2V_0\alpha^2\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(n,l,m)}-2V_0\alpha^2\left\langle\frac{s^{5/2}}{(1-s)^3}\right\rangle_{(n,l,m)}\right)\right\}\quad(45)$$

For the case of spin-0, j equal only one value $j=l$, which allows us to obtain $k(j,l,s)\equiv 0$. Thus the modified energy can be determined according to the following new generalized formula:

$$\begin{aligned}&E^{hmk}_{r-nc}\left(V_0,\alpha,r_e,D_e,n,j=l,s=0,l,m\right)=\mu+E_{nl}+\\&+\left[\Xi\left(E_{nl},n,l,V_0,\alpha,r_e,D_e\right)\aleph\sigma m\right]^{1/2}\end{aligned}\quad(46)$$

On the other hand, it is evident to consider the quantum number m takes ($2l+1$) values and we have also three values for $j=l\pm1,l$, thus any state in ordinary 3-dimensional space of energy for the diatomic molecules with spin-1 under modified Hulthén–Kratzer potential model will become triplet $3(2l+1)$ sub-states. To obtain the total complete degeneracy of energy level of the modified Hulthén–Kratzer potential in the symmetries of (RNC: 3D-RS), we will have to sum for all allowed values of angular momentum quantum number $l=\overline{0,n-1}$. Total degeneracy is thus,

$$\underbrace{2\sum_{l=0}^{n-1}(2l+1)\equiv 2n^2}_{RQM}\rightarrow\begin{cases}\underbrace{3\sum_{l=0}^{n-1}2(2l+1)\equiv 6n^2}_{\text{NCRQM}} & \text{For : spin - 1}\\ \underbrace{\sum_{l=0}^{n-1}2(2l+1)\equiv 2n^2}_{\text{NCRQM}} & \text{For : spin - 0}\end{cases}\quad(47)$$

It is clear that the degeneracy of the initial spectral is automatically broken and replaced by a more precise entity. The triplet of the total complete degeneracy of energy level for the diatomic molecules with spin-1, in RNCQM symmetries under the modified Hulthén–Kratzer potential model, gives very clear physical indicator shows that physical treatments with RNCQM appear more detailed and clarity if compared with similar energy levels obtained in ordinary relativistic quantum mechanics. In order to consider further the interpretation of the positive and negative energy solutions of the MKGE, one can consider the nonrelativistic limit. For this purpose, we apply the following transformations:

$$\begin{aligned}&E^{hmk}_{r-nc}(V_0,\alpha,r_e,D_e,n,j,l,s,m)-\mu\rightarrow E^{hmk}_{nr-nc}(V_0,\alpha,r_e,D_e,n,j,l,s,m)\\&E^{hmk}_{r-nc}(V_0,\alpha,r_e,D_e,n,j,l,s,m)+\mu\rightarrow 2\mu\end{aligned}\quad(48)$$

Here $E^{hmk}_{nr-nc}\left(V_0,\alpha,r_e,D_e,n,j,l,s,m\right)$ is the non-relativistic energy in (NRNC: 3D-RS) symmetries, inserting above transformation into Eqn. (43) yields:

$$\begin{aligned}&E^{hmk}_{nr-nc}\left(V_0,\alpha,r_e,D_e,n,j,l,s,m\right)=E_{nr-nl}-2\mu+\\&+\left[\Xi\left(E_{nl},n,l,V_0,\alpha,r_e,D_e\right)k(l)\left(\Theta+\aleph\sigma m\right)\right]^{1/2}\end{aligned}\quad(49)$$

Where, E_{nr-nl} is the non-relativistic energy in the symmetries of nonrelativistic quantum mechanics and is given in Ref. [8] as follows:

$$E_{nr-nl}=\frac{\alpha^2}{2\mu}\left[2\mu D_e r_e^2+l(l+1)-\frac{4\mu D_e r_e}{\alpha}\right]+D_e-\frac{\alpha^2}{2\mu}\left[\frac{n+\xi}{2}+\frac{2\mu D_e r_e^2+l(l+1)-\frac{4\mu D_e r_e+2\mu A}{\alpha}}{2(n+\xi)}\right]^2\quad(50)$$

Where, $\xi=\frac{1}{2}\left(1+\sqrt{1+4\left(2\mu D_e r_e^2+l(l+1)\right)}\right)$. In the non-relativistic Schrödinger equation Eq. (49) can apply to hydrogen like atoms such as He^{+}, Be^{+} and Li^{2+}, we have $|l-1/2|\leq j\leq|l+1/2|$, allows us to obtain two values ($j=l\pm1/2$) which gives $\left(k_1(l),k_2(l)\right)\equiv\frac{1}{2}(l,-l-1)$ and thus, we obtain two values of the energy shift $\Delta E^{nr}_{hmk}(n,j,l,s,m)$ as follows:

$$\Delta E_{hmk}^{nr}(n,j=l+1/2,l,s,m)=\Xi(E_{nl},n,l,V_0,\alpha,r_e,D_e)\left\{\frac{l}{2}\Theta+B\sigma m\right\} \quad (51)$$
$$\Delta E_{hmk}^{nr}(n,j=l-1/2,l,s,m)=\Xi(E_{nl},n,l,V_0,\alpha,r_e,D_e)\left\{-\frac{l+1}{2}\Theta+B\sigma m\right\}$$

The above results of the degenerate energy shift and Eqn. (38) gives the nonrelativistic energy $E_{nr-nc}^{hmk}(V_0,\alpha,r_e,D_e,n,j,l,s,m)$ of a fermionic particle with - $S=1/2$ under the modified Hulthén–Kratzer potential model:

$$E_{nr-nc}^{hmk}(V_0,\alpha,r_e,D_e,n,j,l,s,m)=E_{nr-nl}+\begin{cases}\left[\Delta E_{hmk}^{nr}(n,j=l+1/2,l,s,m)\right]^{1/2} & \text{for} \quad j=l+1/2\\ \left[\Delta E_{hmk}^{nr}(n,j=l-1/2,l,s,m)\right]^{1/2} & \text{for} \quad j=l-1/2\end{cases} \quad (52)$$

Thus, one can conclude that the MKGE becomes similar to the Duffin–Kemmer equation, which describes bosonic particles with spin non-null. It should be noted that our current results are an excellent agreement with our previously published work and other works in the context of NCQM [10,12,13, 22, 23, 55,57,58]. It is worthwhile to mention that for the two simultaneously limits $(\Theta,\sigma)\to(0,0)$, we recover the results of the commutative space obtained in Ref. [6] For the MHKP model. This means that our present calculations are correct.

5. Conclusions

This section of our paper gives a summary of the basic points in our work. We have investigated the MKGE and MSE for the MHKP model in the relativistic and nonrelativistic non-commutative three-dimensional spaces. The energy $E_{r-nc}^{hmk}(V_0,\alpha,r_e,D_e,n,j,l,m)$ due the non-commutative property corresponding the generalized n^{th} excited states as a function of the shift energy $\Delta E_{hmk}(n,j,l,s,m)$ and E_{nl} due to The MHKP model is obtained via first-order perturbation theory and expressed by the Gamma function, the discreet atomic quantum numbers (j,l,s,m) and the potential parameters (V_0,α,r_e,D_e), in addition to non-commutative two parameters (Θ and σ).

This behavior is similar to the perturbed modified Zeeman Effect, and modified perturbed spin-orbit coupling in which an external magnetic field is applied to the system, and the spin-orbit couplings which are generated with the effect of the perturbed effective potential $V_{pert}^{hmk}(r)$ in the symmetries of relativistic and nonrelativistic non-commutative 3-dimensional real space.

Therefore, we can conclude that the MKGE becomes similar to the Duffin–Kemmer equation under MHKP model, it can describe a dynamic state of a particle with spin one in the symmetries of RNCQM. We have seen that the physical treatment of MKGE under the MHKP model for the diatomic molecules with spin-1 gives a very clear physical indication that physical treatments with RNCQM appear more detailed and clarity if it compared with similar energy levels obtained in ordinary relativistic quantum mechanics. The nonrelativistic limits were treated and the results related to RQM under the Hulthén–Kratzer potential model becomes a particular case when we take simultaneously two limits $(\Theta,\sigma)\to(0,0)$. The comparisons show that our theoretical results are in very good agreement with reported works.

Acknowledgments

This work has been partly supported by the AMHESR under project number B00L02UN280120180001 and by the Laboratory of Physics and Material Chemistry, University of M'sila, Algeria. We thank the reviewers for their helpful criticism and suggestions for valuable improvements to our paper.

References

[1] H. Hulthén, Uber die Eigenlösungen der Schrödinger chung des Deutrons, *Ark. Mat. Astron. Fys. A* **28**, 5 (1942)

[2] M. C. Onyeaju, J. O. A. Idiodi, A. N. Ikot, M. Solaimani and H. Hassanabadi, Linear and Nonlinear Optical Properties in Spherical Quantum Dots: Generalized Hulthén Potential, *Few-Body Syst*. **59**, 793-805 (2016). https://doi.org/10.1007/s00601-016-1110-4

[3] M. Hosseinpour, F. M. Andrade E. O. Silva and H. Hassanabadi, Scattering and bound states for the Hulthén potential in a cosmic string background, *Eur. Phys. J. C* **77**, 270 (2017). https://doi.org/10.1140/epjc/s10052-017-4834-5

[4] O. Bayrak and I. Boztosun, Bound state solutions of the Hulthén potential by using the asymptotic iteration method, *Physica Scripta* **76**(1), 92–96 (2007). doi:10.1088/0031-8949/76/1/016

[5] Y. P.Varshni, Eigenenergies and oscillator strengths for the Hulthén potential, *Phys. Rev. A* **41**, 4682 (1990). DOI: https://doi.org/10.1103/PhysRevA.41.4682

[6] J. A. Obu, P. O. Okoi and U. S. Okorie, Relativistic and nonrelativistic treatment of Hulthén–Kratzer potential model in D-dimensions, *Indian J.Phys* (2019). doi:10.1007/s12648-019-01638-w

[7] O. Bayrak, I. Boztosun and H. Cifti, Exact analytical solutions to the Kratzer potential by the asymptotic iteration method, *Int. J. Quant. Chem.* **107**, 540 (2007). https://doi.org/10.1002/qua.21141

[8] C. O. Edet, U. S. Okorie, A. T. Ngiangia, and A. N. Ikot, Bound state solutions of the Schrodinger equation for the modified Kratzer potential plus screened Coulomb potential, *Indian J.Phys*. (2019). https://doi.org/10.1007/s12648-019-01477-9

[9] C. O. Edet, K. O. Okorie, H. Louis and N. A. Nzeata-Ibe, Any *l*-state solutions of the Schrodinger equation interacting with Hellmann–Kratzer potential model, *Indian J. Phys*. (2019). https://doi.org/10.1007/s12648-019-01467-x

[10] Abdelmadjid Maireche, Any *L*-States Solutions of The Modified Schrodinger Equation with Generalized Hellmann–Kratzer Potential Model in The Symmetries of NRNCQM, *To Physics Journal* **4**, 16-32 (2019). Retrieved from https://purkh.com/index.php/tophy/article/view/521

[11] H. Louis, Benedict I. Ita and N. I. Nelson, K-State Solutions To The Dirac Equation For The Quadratic Exponential-Type Potential Plus Eckart Potential And Coulomb-Like Tensor Interaction Using Nikiforov-Uvarov Method, *To Physics Journal* **3**, 12-23 (2019). Retrieved from https://purkh.com/index.php/tophy/article/view/379

[12] Abdelmadjid Maireche, The Klein–Gordon Equation with Modified Coulomb Potential Plus Inverse-Square–Root Potential in Three-Dimensional Noncommutative Space, *To Physics Journal* **3**, 186-196 (2019). Retrieved from https://purkh.com/index.php/tophy/article/view/489

[13] Abdelmadjid Maireche, The Klein–Gordon equation with modified Coulomb plus inverse-square potential in the noncommutative three-dimensional space, *Modern Physics Letters A* **35**(5) (2020) 2050015. doi:10.1142/s0217732320500157

[14] Abdelmadjid Maireche, A New Model for Describing Heavy-Light Mesons in The Extended Nonrelativistic Quark Model Under a New Modified Potential Containing Cornell, Gaussian And Inverse Square Terms in The Symmetries Of NCQM, *To Physics Journal* **3**, 197-215 (2019). Retrieved from https://purkh.com/index.php/tophy/article/view/500.

[15] H. S. Snyder, Quantized Space-Time, *Phys Rev*. **71** (1947) 38-42 DOI:https://doi.org/10.1103/PhysRev.71.38

[16] S. Capozziello, G. Lambiase and G. Scarpetta, Generalized uncertainty principle from quantum geometry, *Int. J. Theor. Phys*. **39** (2000) 15. https://doi.org/10.1023/A:1003634814685

[17] R. Vilela Mendes, Geometry, stochastic calculus, and quantum fields in a noncommutative space–time, *Journal of Mathematical Physics* **41**(1) (2000) 156–186. doi:10.1063/1.533127

[18] E. Passos, L. R. Ribeiro and C. Furtado , Noncommutative Anandan quantum phase, *Phys Rev A* **76**, 012113, (2007). DOI: https://doi.org/10.1103/PhysRevA.76.012113

[19] L. R. Ribeiro, E. Passos, C. Furtado and J. R. Nascimento, Geometric phases modified by a Lorentz-symmetry violation background, *International Journal of Modern Physics A* **30**(14), 1550072 (2015). Doi: 10.1142/s0217751x15500724

[20] Ö. F. Dayi, Dynamics of dipoles and quantum phases in noncommutative coordinates, *EPL (Euro physics Letters)* **85**(4), 41002 (2009). doi:10.1209/0295-5075/85/41002

[21] Ö. F. Day, and B. Yapışkan, An alternative formulation of Hall effect and quantum phases in noncommutative space, *Physics Letters A* **374**(37), 3810–3817 (2010). doi:10.1016/j.physleta.2010.07.043

[22] H. Motavalli and A. R. Akbaieh, KLEIN–GORDON EQUATION FOR THE COULOMB POTENTIAL IN NONCOMMUTATIVE SPACE, *Modern Physics Letters A*, 25(29), 2523–2528 (2010). doi:10.1142/s0217732310033529

[23] M. Darroodi, H. Mehraban, and H. Hassanabadi, The Klein–Gordon equation with the Kratzer potential in the noncommutative space, *Modern Physics Letters A* **33** No. 35, 1850203 (2018). doi:10.1142/s0217732318502036

[24] Abdelmadjid Maireche, Solutions of Two-dimensional Schrodinger Equation in Symmetries of Extended Quantum Mechanics for the Modified Pseudoharmonic Potential: an Application to Some Diatomic Molecules, *J. Nano- Electron. Phys*. **11** No 4, 04013 (2019). DOI : https://doi.org/10.21272/jnep.11(4).04013

[25] P. Gnatenko, Parameters of noncommutativity in Lie-algebraic noncommutative space, *Physical Review D* **99**(2), 026009-1 (2019). doi:10.1103/physrevd.99.026009

[26] P. Gnatenko, and V. M. Tkachuk, Weak equivalence principle in noncommutative phase space and the parameters of noncommutativity, *Physics Letters A* **381**(31), 2463–2469 (2017). doi:10.1016/j.physleta.2017.05.056

[27] O. Bertolami, J. G. Rosa, C. M. L. De aragao, P. Castorina and D. Zappala, Scaling of varialbles and the relation between noncommutative parameters in noncommutative quantum mechanics, *Modern Physics Letters A* **21**(10), 795–802 (2006). Doi: 10.1142/s0217732306019840

[28] Abdelmadjid Maireche, A Recent Study of Excited Energy Levels of Diatomics for Modified more General Exponential Screened Coulomb Potential: Extended Quantum Mechanics, *J. Nano- Electron. Phys.* **9**(3), 03021 (2017). DOI: 10.21272/jnep.9 (3).03021

[29] E. F. Djemaï, and H. Smail, On Quantum Mechanics on Noncommutative Quantum Phase Space, *Commun. Theor. Phys. (Beijing, China)* **41**(6), 837–844 (2004).

doi:10.1088/0253-6102/41/6/837

[30] Yi YUAN, LI Kang, WANG Jian-Hua and CHEN Chi-Yi, Spin-1/2 relativistic particle in a magnetic field in NC phase space, Chinese Physics C **34**(5) 543–547 (2010). doi:10.1088/1674-1137/34/5/005

[31] O. Bertolami, and P. Leal, Aspects of phase-space noncommutative quantum mechanics, *Physics Letters B* **750,** 6–11 (2015). doi:10.1016/j.physletb.2015.08.024

[32] C. Bastos, O. Bertolami , N. C. Dias and N. JPrata, Weyl–Wigner formulation of noncommutative quantum mechanics, *Journal of Mathematical Physics* **49**(7), 072101 (2008). doi:10.1063/1.2944996

[33] J. Zhang, Fractional angular momentum in non-commutative spaces, *Physics Letters B* **584**(1-2), (2004) 204–209. doi:10.1016/j.physletb.2004.01.049

[34] J. Gamboa, M. Loewe, and J. C. Rojas, Noncommutative quantum mechanics, Phys. Rev. D **64,** 067901 (2001). DOI: https://doi.org/10.1103/PhysRevD.64.067901

[35] M. Chaichian, Sheikh-Jabbari and A.Tureanu , Hydrogen Atom Spectrum and the Lamb Shift in Noncommutative QED, *Physical Review Letters* **86**(13), 2716–2719 (2001). doi:10.1103/physrevlett.86.2716

[36] Abdelmadjid Maireche, New Relativistic Atomic Mass Spectra of Quark (u, d and s) for Extended Modified Cornell Potential in Nano and Plank's Scales, *J. Nano-Electron. Phys*. **8**(1), 01020-1 - 01020-7 (2016). DOI: 10.21272/jnep.8 (1).01020

[37] J. Wang, and K. Li, The HMW effect in noncommutative quantum mechanics, *Journal of Physics A: Mathematical and Theoretical* **40**(9), 2197–2202 (2007). doi:10.1088/1751-8113/40/9/021

[38] Abdelmadjid Maireche, New Bound State Energies for Spherical Quantum Dots in Presence of a Confining Potential Model at Nano and Plank's Scales, *NanoWorld J.* **1**(4), 122-129 (2016). Doi: 10.17756/nwj.2016-016

[39] K. Li, and J. Wang, The topological AC effect on non-commutative phase space, *The European Physical Journal C.* **50**(4) (2007) 1007–1011. doi:10.1140/epjc/s10052-007-0256-0

[40] Abdelmadjid Maireche, A Complete Analytical Solution of the Mie-Type Potentials in Non-commutative 3-Dimensional Spaces and Phases Symmetries, *Afr. Rev Phys*. **11,** 111-117 (2016).

[41] Abdelmadjid Maireche, A New Nonrelativistic Investigation for the Lowest Excitations States of Interactions in One-Electron Atoms, Muonic, Hadronic and Rydberg Atoms with Modified Inverse Power Potential, *International Frontier Science Letters* **9**, 33-46 (2016). DOI: https://doi.org/10.18052/www.scipress.com/IFSL.9.33

[42] Abdelmadjid Maireche, New quantum atomic spectrum of Schrödinger equation with pseudo harmonic potential in both noncommutative three-dimensional spaces and phases, *Lat. Am. J. Phys. Educ*. **9**(1) (2015)1301.

[43] Abdelmadjid Maireche, New Bound States for Modified Vibrational-Rotational Structure of Supersingular plus Coulomb Potential of the Schrödinger Equation in One-Electron Atoms, *International Letters of Chemistry, Physics and Astronomy* **73,** 31-45 (2017). DOI: https://doi.org/10.18052/www.scipress.com/ILCPA.73.31

[44] Abdelmadjid Maireche, Extended of the Schrödinger Equation with New Coulomb Potentials plus Linear and Harmonic Radial Terms in the Symmetries of Noncommutative Quantum Mechanics, *J. Nano-Electron. Phys.* **10**(6), 06015-1 - 06015-7 (2018). DOI: https://doi.org/10.21272/jnep.10(6).06015

[45] Abdelmadjid Maireche, Investigations on the Relativistic Interactions in One-Electron Atoms with Modified Yukawa Potential for Spin 1/2 Particles, *International Frontier Science Letters* **11,** (2017) 29. DOI: https://doi.org/10.18052/www.scipress.com/IFSL.11.29

[46] Abdelmadjid Maireche, New Nonrelativistic Three-Dimensional Spectroscopic Studies of NMGECSC Potential in Presence of External Electric, *J. Nano-Electron. Phys.* **10** No 4, 04003 (2018). http://dx.doi.org/10.21272/jnep.10(4).04003

[47] Abdelmadjid Maireche, Effects of Three-Dimensional Noncommutative Theories on Bound States Schrödinger Molecular under New Modified Kratzer-type Interactions, *J. Nano- Electron. Phys.* **10** No 2, 02011 (2018). https://doi.org/10.21272/jnep.10(2).02011

[48] M. A. De Andrade and C. Neves, Noncommutative mapping from the symplectic formalism, *Journal of Mathematical Physics* **59**(1), 012105 (2018). doi:10.1063/1.4986964

[49] E. M. C. Abreu, C. Neves and W. Oliveira, Noncommutativity from the symplectic point of view, *Int. J. Mod. Phys. A* **21,** 5359 (2006). doi:10.1142/s0217751x06034094

[50] E. M. C. Abreu, J. A. Neto, A. C. R. Mendes, C. Neves, W. Oliveira and M. V. Marcial, Lagrangian formulation for noncommutative nonlinear systems, *Int. J. Mod. Phys. A*. **27,** 1250053 (2012). doi:10.1142/s0217751x12500534

[51] L. Mezincescu, Star operation in quantum mechanics, e-print arXiv: hep-th/0007046v2.

[52] R. Khordad, and H. R Rastegar Sedehi, Magnetic susceptibility of graphene in non-commutative phase-space: Extensive and non-extensive entropy, *The European Physical Journal Plus* **134**(4) (2019). doi:10.1140/epjp/i2019-12558-5

[53] R. L. Greene and C. Aldrich, Variational wave functions for a screened Coulomb potential, *Physical Review A* **14**(6), 2363–2366 (1976). doi:10.1103/physreva.14.2363

[54] S. H. Dong, W. C. Qiang, G. H.Sun and V. B. Bezerra,Analytical approximations to the l-wave solutions of the Schrödinger equation with the Eckart potential, *Journal of Physics A: Mathematical and Theoretical* **40**(34), 10535–10540 (2007). doi:10.1088/1751-8113/40/34/010

[55] M. Alberg and L. Wilets, Exact solutions to the Schrödinger equation for potentials with Coulomb and harmonic oscillator terms, *Physics Letters A* **286**(1), 7–14 (2001). Doi: 10.1016/s0375-9601(01)00385-1

[56] S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products, 7th. Ed.; Elsevier, edited by Alan Jeffrey (University of Newcastle upon Tyne, England)

and Daniel Zwillinger (Rensselaer Polytechnic Institute USA) 2007

[57] Abdelmadjid Maireche, A New Asymptotic Study to the 3-Dimensional Radial Schrödinger Equation under Modified Quark-antiquark Interaction Potential, *J Nanosci Curr Res* **4**(1), 131 (2019).

[58] Abdelmadjid Maireche, Nonrelativistic treatment of Hydrogen-like and neutral atoms subjected to the generalized perturbed Yukawa potential with centrifugal barrier in the symmetries of noncommutative Quantum mechanics, *International Journal of Geometric Methods in Modern Physics.* https://doi.org/10.1142/S021988782050067X

Received: 02 December, 2019

Accepted (revised version): 27 May, 2020