Iso-spectral Instability of Harmonic Oscillator: Breakdown of Unbroken Pseudo-Hermiticity and $\mathcal{PT}$ Symmetry Condition

Biswanath Rath*

Department of Physics, North Orissa University, Takatpur, Baripada, Odisha, India

We propose a direct approach to study spectral stability and instability for general harmonic oscillator under non-Hermitian transformation. Interestingly, we notice that spectral instability becomes an inherent feature in non-Hermitian transformation of coordinate, momentum or both. The instability in spectrum is due to the appearance of complex energy levels in higher quantum states reflecting the breakdown of Pseudo-Hermiticity and $\mathcal{PT}$ symmetry condition, provided the frequency of oscillation of transformed Hamiltonian matches with that of variational calculation i.e., $w = w_v > 0$. However, spectral stability is associated with the frequency of oscillation matching with that of perturbative calculation.

1. Introduction

In quantum physics, the study of spectral stability started with only two simple but exactly solvable Hamiltonians i.e., Hydrogen atom and harmonic oscillator [1]. These two Hamiltonians are Hermitian in nature and having a wide range of application in physics. The basic understanding of quantum mechanics is that Hermitian operators can yield real energy eigen-values [1]. More precisely, operators having self-adjoint behavior must yield real spectrum [1], However, Hermiticity cannot explain the spectra of cubic oscillator

$$H = p^2 + x^3 \quad (1)$$

or

$$H = p^2 + x^2 + x^3 \quad (2)$$

On the other hand complex cubic oscillators

$$H = p^2 + ix^3 \quad (3)$$

and

$$H = p^2 + x^2 + ix^3 \quad (4)$$

yield real energy eigen-values [2,3]. The existence of real energy eigen-values are due to **PT** invariance condition as proposed by Bender and Boettecher [2] (hence forward BB). However, in the work of BB [2] Hamiltonians containing the term [4]

$$f(x)\, f(p) + f(p)\, f(x) \quad (5)$$

did not find a place, which promoted Mostafazadeh [5] to propose a new pseudo-Hermiticity condition

$$\eta H \eta^{-1} = H^{+} \quad (6)$$

in place of $\mathcal{PT}$ invariance condition [2] i.e.,

$$[H, \mathcal{PT}] = 0 \quad (7)$$

As such we have only three eigen-value conditions (Hermiticity $H = H^{+}$, $\mathcal{PT}$ symmetry and pseudo-Hermiticity). An interesting revelation on Eqns. (3) and (4) started when Krejerik and Siegl (hence forward KS) [6] reported pseudo-spectrum, which can be visualized by making a transformation in real space ($x \rightarrow x+\lambda$). According to KS [6], a small perturbation can make a drastic change in the spectra of complex cubic oscillators. At this point a reader will notice that after real transformation the original Hamiltonian in Eqn. (3) or Eqn. (4) is not $\mathcal{PT}$ symmetric. Hence the comparison on energy levels of Hamiltonians before and after transformation is not justified. However in order to study the spectral instability, Rath, Mallick and Samal [7] (hence forward RMS) proposed the idea of complex transformation of co-ordinate ($x \rightarrow x - i\lambda$) with a view to make the Hamiltonians to preserve the $\mathcal{PT}$ invariance condition. It is worth mentioning the findings of RMS [7], which reflect the instability in spectra of complex cubic oscillators. In other words, this conclusion clearly demonstrates the breakdown of energy eigen-value condition of $\mathcal{PT}$ symmetry as proposed earlier [2].

*biswanathrath10@gmail.com

Further review of literature on spectral stability reflects another interesting problem relating to simple harmonic oscillator under a non-Hermitian transformation of momentum [8]

$$H_\beta = \frac{(p+i\beta x)}{2} + \left(\alpha^2 + \beta^2\right)\frac{x^2}{2} \quad (8)$$

Where the author has proved only iso-spectral behavior with the harmonic oscillator

$$H_{SHO} = \frac{p^2}{2} + (\alpha^2 + \beta^2)\frac{x^2}{2} \quad (9)$$

In this case, one will notice that the commutation relation,

$$[x, p + i\,\beta\, x] = [x, p] = i \quad (10)$$

remains invariant. A simplification of the above equation reflects that

$$H_\beta = \frac{1}{2}[p^2 + \alpha^2 x^2 + i\beta\,(xp + px)] \quad (11)$$

Hamiltonian is not only **PT** symmetric but also pseduo-Hermitian in nature. The question arises whether the Hamiltonian in Eqn. (8) can always admit real energy levels.

The answer to this question is in negative. Hence, the aim of this paper is to generate non-Hermitian terms in general harmonic oscillator of the type

$$H_{AB} = A^2p^2 + B^2x^2 \quad (12)$$

and study its spectral stability or instability by making a comparison with the energy eigen-values

$$\epsilon_n = AB\,(2n + 1) \quad (13)$$

using a direct approach.

2. Non-Hermitian Harmonic Oscillator Hamiltonian and Iso-spectrum or Spectral Stability

Let us make non-Hermitian transformation [9] of co-ordinate of x as

$$x \to \frac{x+iRp}{\sqrt{1+LR}} \quad (14)$$

and momentum p as

$$p \to \frac{p+iLx}{\sqrt{1+LR}} \quad (15)$$

in the Hamiltonian of harmonic oscillator given in Eqn. (12). In the above choice of transformation, we notice that the invariance of commutation relation

$$[x, p] = [x + iRp, p + iLx] = i \quad (16)$$

is preserved.

The transformed Hamiltonian is

$$H = \frac{1}{1+LR}[A^2(p + iLx)^2 + B^2(x + iRp)^2] \quad (17)$$

Now, let us assume that the frequency of oscillation of transformed oscillator (Eqn. (13)) is w = AB > 0, which appears in the wave function as

$$|n\ >_w = [\frac{\sqrt{w}}{\sqrt{\pi n! 2^n}}]^{1/2} H_n\,(\sqrt{wx}) e^{\frac{-wx^2}{2}} \quad (18)$$

2.1. Determination of w from the condition $< \boldsymbol{n}|\boldsymbol{H}|\ \boldsymbol{n} + \mathbf{2} >_{\boldsymbol{w}} = \mathbf{0}$

Using standard matrix element calculation [10]

$$< n|H|n + 2 >_w = 0 \quad (19)$$

we get

$$w_1 = \frac{AL+B}{A-BR} \quad (20)$$

and

$$w_2 = \frac{AL-B}{A+BR} \quad (21)$$

2.2. Determination of w from the condition $< n + 2|\boldsymbol{H}|\boldsymbol{n} >_{\boldsymbol{w}} = \mathbf{0}$

Similarly using the condition

$$< n + 2|H|n >_w = 0 \quad (22)$$

we get

$$w_3 = \frac{B-AL}{A+BR} \quad (23)$$

3. Spectral Instability: w = w_v from Variational Method

Now, we use the standard variational method [10] to determine the frequency of oscillation from diagonal term as follows

$$\frac{d<n|H|n>_w}{dw} = 0 \quad (24)$$

Here we get an interesting relation on $w = w_v$, where

$$w = w_v = \sqrt{\frac{B^2 - L^2A^2}{A^2 - R^2B^2}} \quad (25)$$

In the direct study approach we use different values of w and calculate energy eigen-values.

4. Direct Calculation of Energy and Commutation Relation. Using MATLAB

In the direct study, we use MATLAB program, which is a modified version [12] for easy calculation of energy. This can also be used for visualizing invariance in commutation relation. The MATLAB program is given below.

B-MATLAB

$$N = 100 \quad (26)$$

$$s = 1; \quad (27)$$

$$n = 1 : N - 1; \quad (28)$$

$$m = \sqrt{(n)}; \quad (29)$$

$$A = 1; \quad (30)$$

$$B = 3; \quad (31)$$

$$W = 3; \quad (32)$$

$$w = W; \quad (33)$$

$$L = 3; \quad (34)$$

$$R = 4; \quad (35)$$

$$x = \frac{s}{\sqrt{\frac{2}{w}}}\left(\text{diag}\,(m, -1) + \text{diag}\,(m, 1)\right); \quad (36)$$

$$p = \frac{i}{s\sqrt{\frac{2}{w}}}\left(\text{diag}\,(m, -1) - \text{diag}\,(m, 1)\right); \quad (37)$$

$$C = \frac{1}{1 + LR} \quad (38)$$

$$y = (p + iLx); \quad (39)$$

$$z = (x + iRp); \quad (40)$$

$$H = \frac{C\,(zy - yz)}{i}; \quad (41)$$

$$H = C\,(A^2 y^2 + B^2 z^2) \quad (42)$$

$$\text{Eig Sort} = \text{sort}\left(\text{eig}\,(H)\right); \quad (43)$$

$$\text{Eig Sort}\,(1 : 50) \quad (44)$$

One can change the value of w and study the eigen-values. Similarly, one can use it to verify the commutation relation.

5. Result and Discussion

In Table 1a, we show the iso-spectral behavior for different values of L, W ($A = 1$; $B^2 = L^2 + W^2$; $R = 0$) using the appropriate frequency of oscillation $w = \sqrt{W^2 + L^2} \mp L$. In Table 1b, we show the instability in the iso-spectral behavior for same values of B, L, W ($A = 1$; $B^2 = L^2 + W^2$; $R = 0$) using the frequency of oscillation $w = w_v = W$. The interesting point in this selection is to find out spectral instability in non-Hermitian Hamiltonian previously proposed by Ahmed [7], where the author claimed only spectral stability. In Table 2a, we show the iso-spectral behavior for different values of R, W ($B = 1$; $A^2 = R^2 + W^2$; $L = 0$) using the appropriate frequency of oscillation $w = \frac{1}{A-R}$. In Table 2b, we show the instability in iso-spectral behavior for same values of R, W ($B = 1$; $A^2 = R^2 + W^2$; $L = 0$) using the frequency of oscillation $w = w_v = 1/W$.

In Table 3a, we show the iso-spectral behavior for ($B = A = 1$; $W = R = 0$) different values of L using the appropriate frequency of oscillation $w = 1+L$. In Table 3b, we show the instability in iso-spectral behavior for for the same values of L ($B = A = 1$) using the frequency of oscillation $w = w_v = \sqrt{1 - L^2}$.

In Table 4a we show the iso-spectral behavior for ($B = A = 1$; $W = L = 0$) for different values of R using the appropriate frequency of oscillation $\frac{1}{1-R}$. In Table 4b, we show the instability in iso-spectral behavior for ($B = A = 1$) for the same values of R using the frequency of oscillation $w = w_v = \frac{1}{\sqrt{1-R^2}}$.

In Table 5a we show iso-spectral behavior under simultaneous transformation ($A = B = 5$; $R = L = 0.5$) for different values of w using the appropriate frequency of oscillation $w = \frac{B \mp AL}{A \mp BR}$. In Table 5b, we show the instability in iso-spectral behavior for ($A = B = 5$; $L = R = 0.5l$) the frequency of oscillation $w = w_v = \sqrt{\frac{B^2 - A^2L^2}{A^2 - B^2R^2}}$. Similarly, one can consider different values of A, B, L, W, and R, and verify the instability in spectral nature by using the frequency of oscillation $w = w_v$. In conclusion, we notice that if the frequency of oscillation is $w = w_v$, the spectrum contains complex eigen-values for which both $\mathcal{PT}$ symmetry and pseudo-Hermiticity condition breaks down.

Table 1a: Iso-spectra stability of H with A=1, R=0 and B=$\sqrt{W^2 + L^2}$

W	L	w=B±L	E_n-H	$\in_n$	Remarks
4	3	B+L	5	5	iso-spectra
4	3	B+L	15	15	iso-spectra
4	3	B+L	25	25	iso-spectra
4	3	B+L	35	35	iso-spectra
4	3	B+L	45	45	iso-spectra
4	3	B+L	55	55	iso-spectra
4	3	B+L	445	445	iso-spectra
4	3	B+L	455	455	iso-spectra
4	3	B+L	465	465	iso-spectra
4	3	B+L	475	475	iso-spectra
4	3	B+L	485	485	iso-spectra
4	3	B+L	495	495	iso-spectra
3	4	B-L	5	5	iso-spectra
3	4	B-L	15	15	iso-spectra
3	4	B-L	25	25	iso-spectra
3	4	B-L	35	35	iso-spectra
3	4	B-L	45	45	iso-spectra
3	4	B-L	55	55	iso-spectra
3	4	B-L	445	445	iso-spectra
3	4	B-L	455	455	iso-spectra
3	4	B-L	465	465	iso-spectra
3	4	B-L	475	475	iso-spectra
3	4	B-L	485	485	iso-spectra
3	4	B-L	495	495	iso-spectra

Table 1b: Instability in iso-spectra of H with A=1, R=0 and B=$\sqrt{W^2 + L^2}$

W	L	w=W	$E_n \rightarrow$ H	$\in_n$	Remarks
4	3	W	5	5	iso-spectra
4	3	W	15	15	iso-spectra
4	3	W	25	25	iso-spectra
4	3	W	35	35	iso-spectra
4	3	W	45	45	iso-spectra
4	3	W	55	55	iso-spectra
4	3	W	395.53-59.95i	445	No iso-spectra
4	3	W	395.53+59.95i	455	No iso-spectra
4	3	W	398.30-50.18i	465	No iso-spectra
4	3	W	398.30+50.18i	475	No iso-spectra
4	3	W	412.52-82.47i	485	No iso-spectra
4	3	W	412.52+82.47i	495	No iso-spectra
3	4	W	5	5	iso-spectra
3	4	W	15	15	iso-spectra
3	4	W	25	25	iso-spectra
3	4	W	35	35	iso-spectra
3	4	W	45	45	iso-spectra
3	4	W	55	55	iso-spectra
3	4	W	281.71-137.20 i	445	No iso-spectra
3	4	W	281.71+137.20 i	455	No iso-spectra
3	4	W	280.41-144.83 i	465	No iso-spectra
3	4	W	280.41+144.83 i	475	No iso-spectra
3	4	W	295.72-163.26 i	485	No iso-spectra
3	4	W	295.72+163.26 i	495	No iso-spectra

Table 2a: Iso- spectra stability of H with B=1, L=0 and A=$\sqrt{W^2 + R^2}$

W	R	$W=\frac{1}{A-R}$	$E_n \rightarrow H$	ϵ_n	Remarks
4	3	W	5	5	iso-spectra
4	3	W	15	15	iso-spectra
4	3	W	25	25	iso-spectra
4	3	W	35	35	iso-spectra
4	3	W	45	45	iso-spectra
4	3	W	55	55	iso-spectra
4	3	W	445	445	iso-spectra
4	3	W	455	455	iso-spectra
4	3	W	465	465	iso-spectra
4	3	W	475	475	iso-spectra
4	3	W	485	485	iso-spectra
4	3	W	495	495	iso-spectra
3	4	W	5	5	iso-spectra
3	4	W	15	15	iso-spectra
3	4	W	25	25	iso-spectra
3	4	W	35	35	iso-spectra
3	4	W	45	45	iso-spectra
3	4	W	55	55	iso-spectra
3	4	W	445	445	iso-spectra
3	4	W	455	455	iso-spectra
3	4	W	465	465	iso-spectra
3	4	W	475	475	iso-spectra
3	4	W	485	485	iso-spectra
3	4	W	495	495	iso-spectra

Table 2b: Instability in iso-spectra of H with B=1,L=0 and A=$\sqrt{W^2 + R^2}$

W	R	$W=\frac{1}{W}$	$E_n \rightarrow H$	ϵ_n	Remarks
4	3	W	5	5	iso-spectra
4	3	W	15	15	iso-spectra
4	3	W	25	25	iso-spectra
4	3	W	35	35	iso-spectra
4	3	W	45	45	iso-spectra
4	3	W	55	55	iso-spectra
4	3	W	395.53-59.95i	445	No iso-spectra
4	3	W	395.53+59.95i	455	No iso-spectra
4	3	W	398.30-50.18i	465	No iso-spectra
4	3	W	398.30+50.18i	475	No iso-spectra
4	3	W	412.52-82.47i	485	No iso-spectra
4	3	W	412.52+82.47i	495	No iso-spectra
3	4	W	5	5	iso-spectra
3	4	W	15	15	iso-spectra
3	4	W	25	25	iso-spectra
3	4	W	35	35	iso-spectra
3	4	W	45	45	iso-spectra
3	4	W	55	55	iso-spectra
3	4	W	281.71-137.20 i	445	No iso-spectra
3	4	W	281.71+137.20 i	455	No iso-spectra
3	4	W	280.41-144.83 i	465	No iso-spectra
3	4	W	280.41+144.83 i	475	No iso-spectra
3	4	W	295.72-163.26 i	485	No iso-spectra
3	4	W	295.72+163.26 i	495	No iso-spectra

Table 3a: Iso-spectra stability of H with A=B=1, R=0

L	W=I+L	$E_n \to H$	$\in_n$	Remarks
0.6	I+L	1	1	iso-spectra
0.6	I+L	3	3	iso-spectra
0.6	I+L	5	5	iso-spectra
0.6	I+L	7	7	iso-spectra
0.6	I+L	9	9	iso-spectra
0.6	I+L	11	11	iso-spectra
0.6	I+L	89	89	iso-spectra
0.6	I+L	91	91	iso-spectra
0.6	I+L	93	93	iso-spectra
0.6	I+L	95	95	iso-spectra
0.6	I+L	97	97	iso-spectra
0.6	I+L	99	99	iso-spectra
0.8	I+L	1	1	iso-spectra
0.8	I+L	3	3	iso-spectra
0.8	I+L	5	5	iso-spectra
0.8	I+L	7	7	iso-spectra
0.8	I+L	9	9	iso-spectra
0.8	I+L	11	11	iso-spectra
0.8	I+L	89	89	iso-spectra
0.8	I+L	91	91	iso-spectra
0.8	I+L	93	93	iso-spectra
0.8	I+L	95	95	iso-spectra
0.8	I+L	97	97	iso-spectra
0.9	I+L	99	99	iso-spectra

Table 3b: Instability in iso-spectra of H with A=B+1, R=0

L	$W=\sqrt{I-L^2}$	$E_n \to H$	$\in_n$	Remarks
0.6	W	1	1	iso-spectra
0.6	W	3	3	iso-spectra
0.6	W	5	5	iso-spectra
0.6	W	7	7	iso-spectra
0.6	W	9	9	iso-spectra
0.6	W	11	11	iso-spectra
0.6	W	79.1055-11.9897 i	89	No iso-spectra
0.6	W	79.1055+11.9897 i	91	No iso-spectra
0.6	W	79.6603-10.0365 i	93	No iso-spectra
0.6	W	79.6603+10.0365 i	95	No iso-spectra
0.6	W	82.5033-16.4932 i	97	No iso-spectra
0.6	W	82.5033+16.4932 i	99	No iso-spectra
0.8	W	1	1	iso-spectra
0.8	W	3	3	iso-spectra
0.8	W	5	5	iso-spectra
0.8	W	7	7	iso-spectra
0.8	W	9	9	iso-spectra
0.8	W	11	11	iso-spectra
0.8	W	56.3426-27.4401 i	89	No iso-spectra
0.8	W	56.3426+27.4401 i	91	No iso-spectra
0.8	W	56.0820-28.9669 i	93	No iso-spectra
0.8	W	56.0820+28.9669 i	95	No iso-spectra
0.8	W	59.1437-32.6529	97	No iso-spectra
0.8	W	59.1437+32.6529	99	No iso-spectra

Table 4a: Iso spectra stability of H with A=B=1, L=0

R	$W=\frac{1}{I-R}$	$E_n \to H$	$\in_n$	Remarks
0.6	W	1	1	iso-spectra
0.6	W	3	3	iso-spectra
0.6	W	5	5	iso-spectra
0.6	W	7	7	iso-spectra
0.6	W	9	9	iso-spectra
0.6	W	11	11	iso-spectra
0.6	W	89	89	iso-spectra
0.6	W	91	91	iso-spectra
0.6	W	93	93	iso-spectra
0.6	W	95	95	iso-spectra
0.6	W	97	97	iso-spectra
0.6	W	99	99	iso-spectra
0.8	W	1	1	iso-spectra
0.8	W	3	3	iso-spectra
0.8	W	5	5	iso-spectra
0.8	W	7	7	iso-spectra
0.8	W	9	9	iso-spectra
0.8	W	11	11	iso-spectra
0.8	W	89	89	iso-spectra
0.8	W	91	91	iso-spectra
0.8	W	93	93	iso-spectra
0.8	W	95	95	iso-spectra
0.8	W	97	97	iso-spectra
0.9	W	99	99	iso-spectra

Table 4b: Instability in iso-spectra of H with A=B=1, L=0

R	$w=\frac{1}{\sqrt{1-R^2}}$	$E_n \to H$	$\in_n$	Remarks
0.6	w	1	1	iso-spectra
0.6	w	3	3	iso-spectra
0.6	w	5	5	iso-spectra
0.6	w	7	7	iso-spectra
0.6	w	9	9	iso-spectra
0.6	w	11	11	iso-spectra
0.6	w	79.1055-11.9897 i	89	No iso-spectra
0.6	w	79.1055+11.9897 i	91	No iso-spectra
0.6	w	79.6603-10.0365 i	93	No iso-spectra
0.6	w	79.6603+10.0365 i	95	No iso-spectra
0.6	w	82.5033-16.4932 i	97	No iso-spectra
0.6	w	82.5033+16.4932 i	99	No iso-spectra
0.8	w	1	1	iso-spectra
0.8	w	3	3	iso-spectra
0.8	w	5	5	iso-spectra
0.8	w	7	7	iso-spectra
0.8	w	9	9	iso-spectra
0.8	w	11	11	iso-spectra
0.8	w	56.3426-27.4401 i	89	No iso-spectra
0.8	w	56.3426+27.4401 i	91	No iso-spectra
0.8	w	56.0820-28.9669 i	93	No iso-spectra
0.8	w	56.0820+28.9669 i	95	No iso-spectra
0.8	w	59.1437-32.6529 i	97	No iso-spectra
0.8	w	59.1437+32.6529 i	99	No iso-spectra

Table 5a: Iso-spectral stability of H under simultaneous transformation

A=B	R=L	$w=\frac{B\pm AL}{A\mp BR}$	$E_n \rightarrow H$	ϵ_n	Remarks
5	0.5	w	25	25	iso-spectra
5	0.5	w	75	75	iso-spectra
5	0.5	w	125	125	iso-spectra
5	0.5	w	175	175	iso-spectra
5	0.5	w	225	225	iso-spectra
5	0.5	w	275	275	iso-spectra
5	0.5	w	2225	2225	iso-spectra
5	0.5	w	2275	2275	iso-spectra
5	0.5	w	2325	2325	iso-spectra
5	0.5	w	2375	2375	iso-spectra
5	0.5	w	2425	2425	iso-spectra
5	0.5	w	2475	2475	iso-spectra

Table 5b: Iso-spectral instability of H under simultaneous transformation

A=B	R=L	$w = \sqrt{\frac{B^2 - L^2A^2}{A^2 - R^2B^2}}$	$E_n \rightarrow H$	ϵ_n	Remarks
5	0.5	w	25	25	iso-spectra
5	0.5	w	75	75	iso-spectra
5	0.5	w	125	125	iso-spectra
5	0.5	w	175	175	iso-spectra
5	0.5	w	225	225	iso-spectra
5	0.5	w	275	275	iso-spectra
5	0.5	w	1408.6-686.0 i	2225	No iso-spectra
5	0.5	w	1408.6+686.0 i	2275	No iso-spectra
5	0.5	w	1402.0-724.2 i	23252325	No iso-spectra
5	0.5	w	1402.0+724.0 i	23752375	No iso-spectra
5	0.5	w	1478.6-816.3 i	2425	No iso-spectra
5	0.5	w	1478.6+816.3 i	2475	No iso-spectra

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